Range searching with Range Trees

1D

- Balanced BinarySearchTree
  - Build: O(n lg n)
  - Space: O(n)
  - Range queries: O(lg n +k)

2D

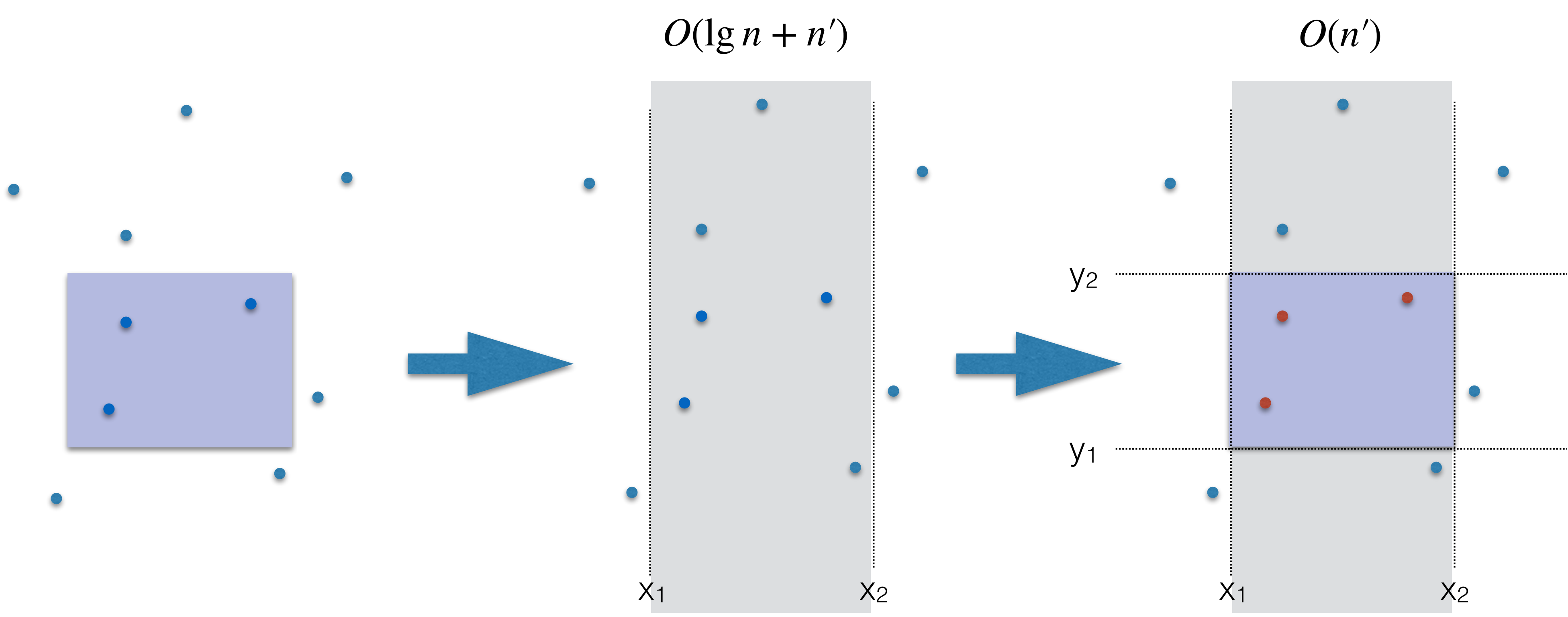
- kd-trees
  - Build: O(n lg n)
  - Space: O(n)
  - Range queries: $O(\sqrt{n} + k)$

- Range trees
  - Build: O(n lg n)
  - Space: O(n lg n)
  - Range queries: O(lg n + k)

Different trade-offs

Towards range trees

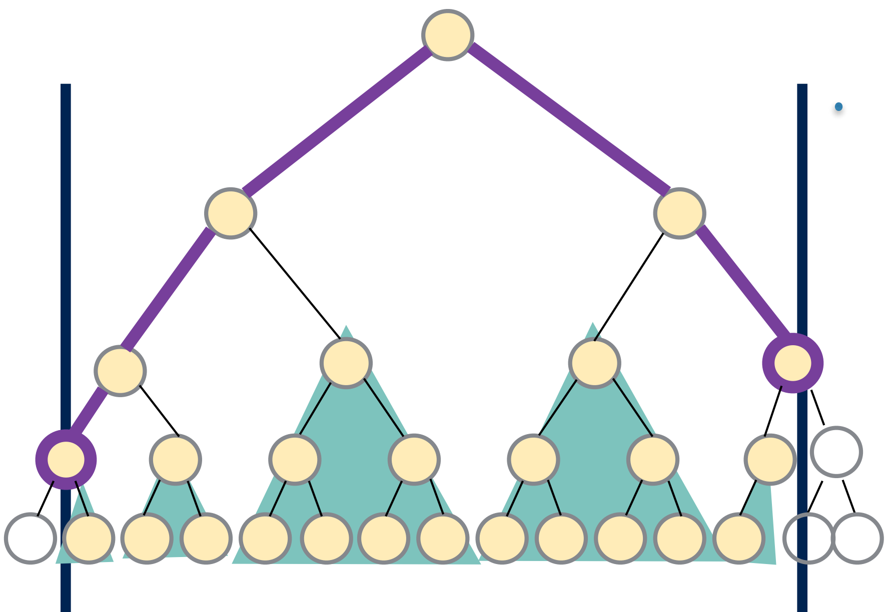
- Build BBST by x-coord
- Range queries: find all points in $[x_1, x_2] \times [y_1, y_2]$
  - Use BBST to find all points with the x-coordinates in $[x_1, x_2]$
  - Traverse these points and find those with y-coord in $[y_1, y_2]$



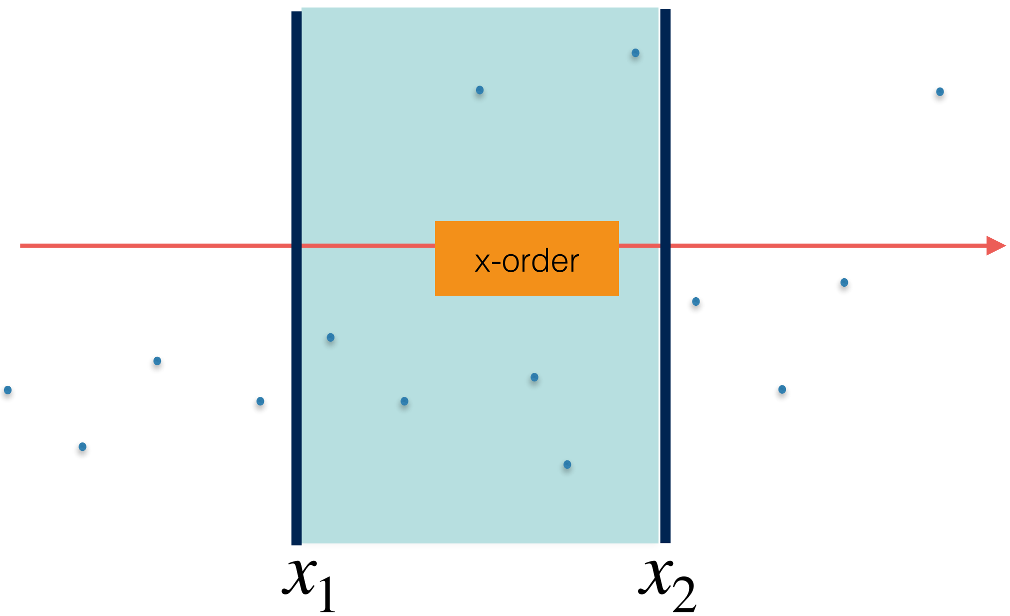
Slow if n' is large but k is small

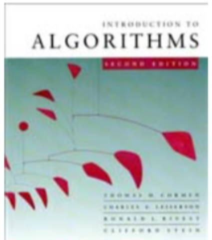
A closer look

- Use BBST to find all points with the x-coordinates in [x1, x2]

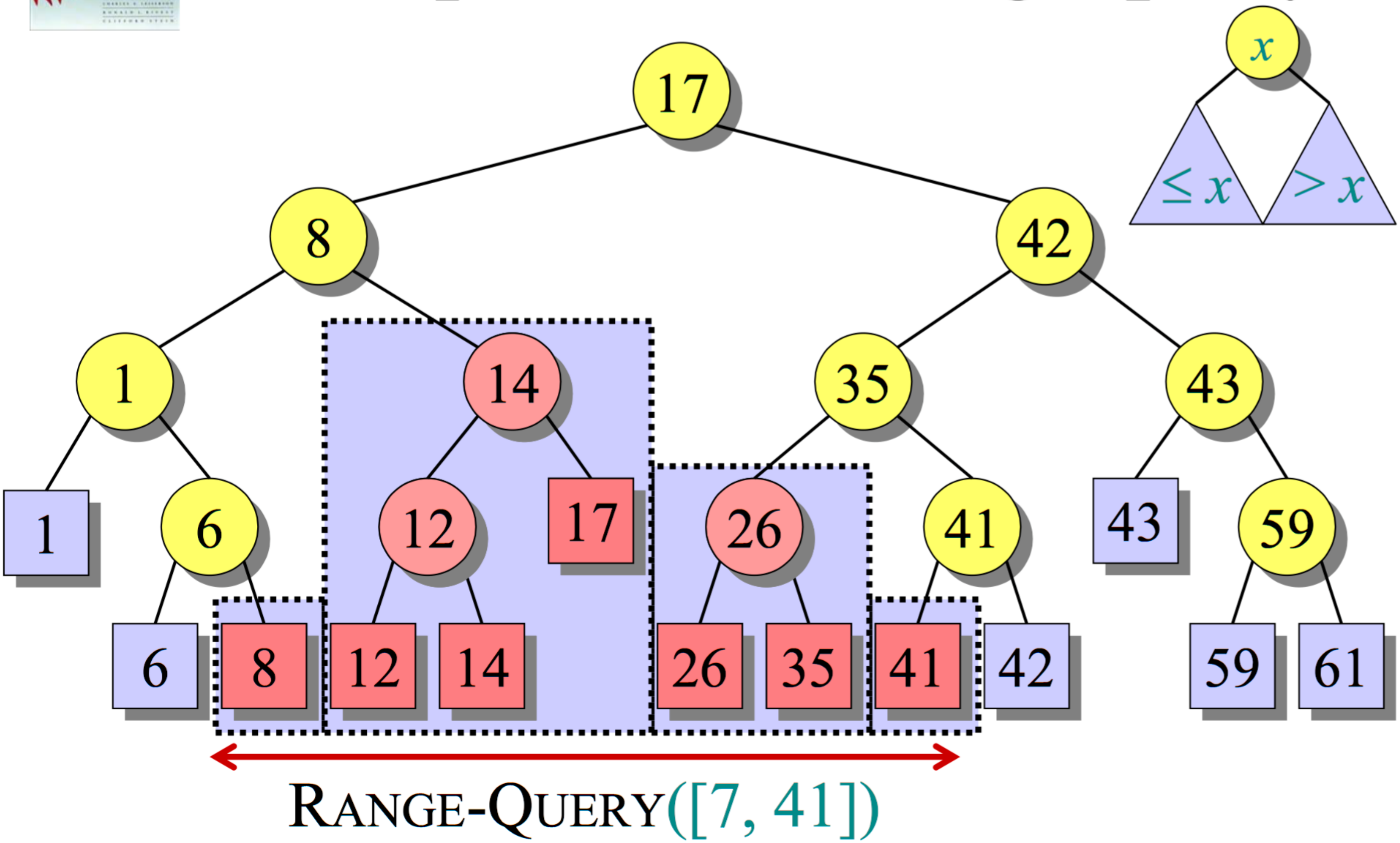


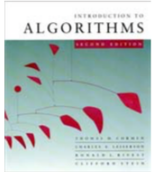
BBST by x-coord



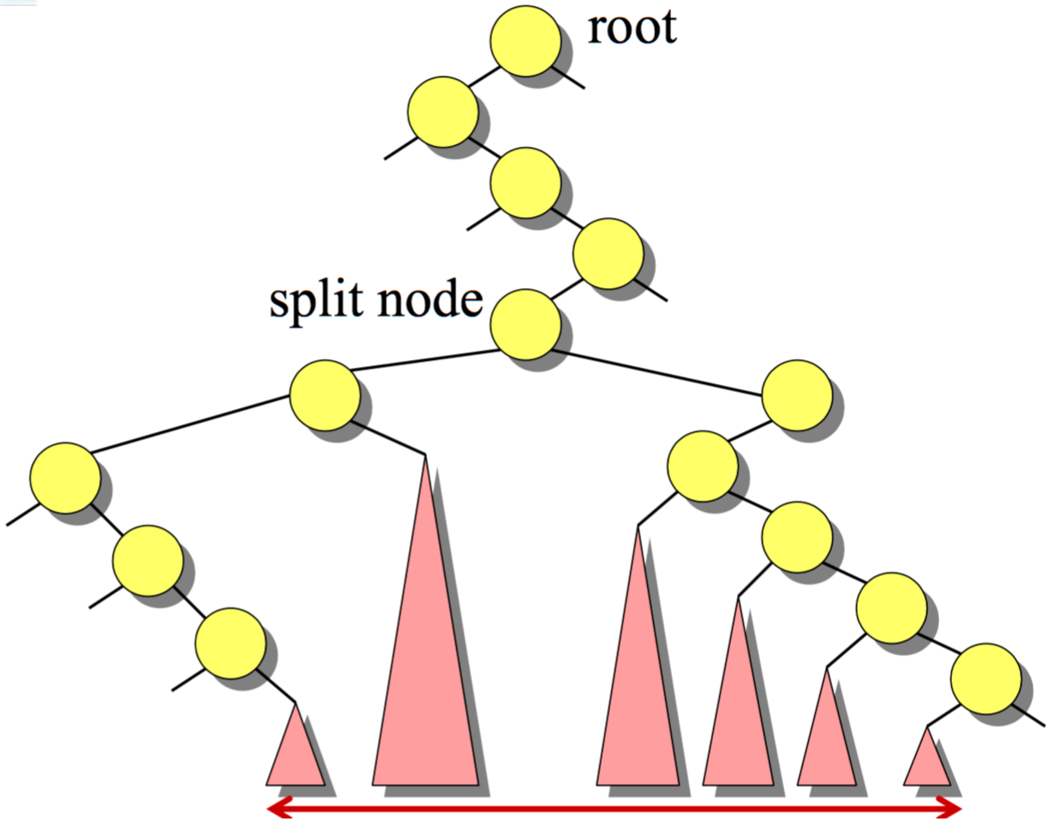


Example of a 1D range query





General 1D range query

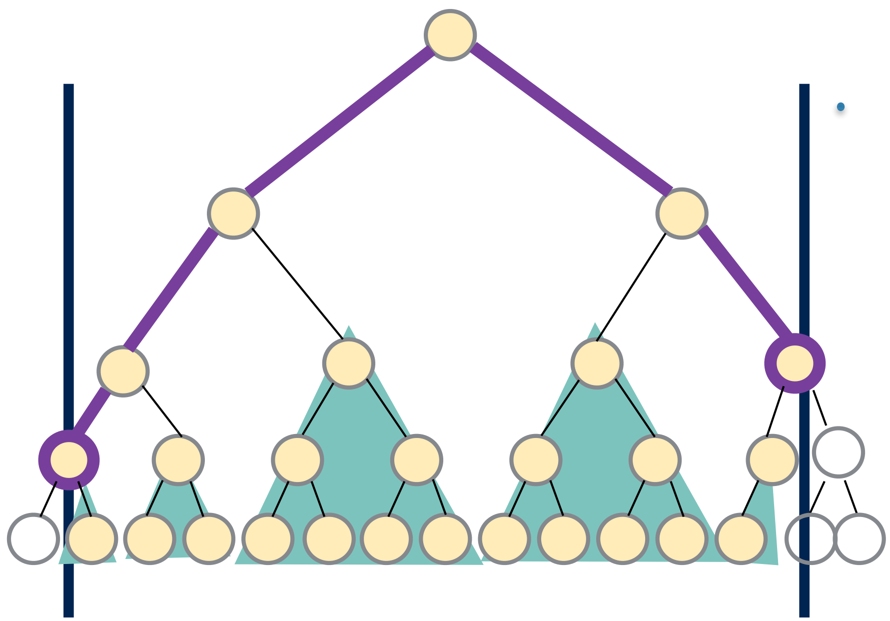


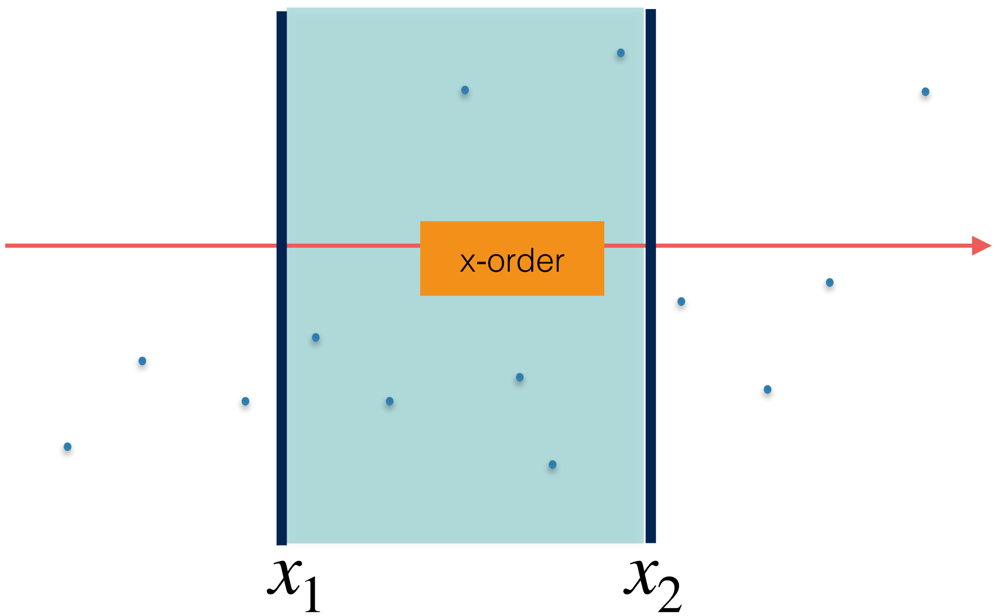
The k points in the range sit in $O(\lg n)$ subtrees

A closer look

- Use BBST to find all points with the x-coordinates in [x1, x2]

The points in $[x_1, x_2]$ sit in $O(\lg n)$ subtrees





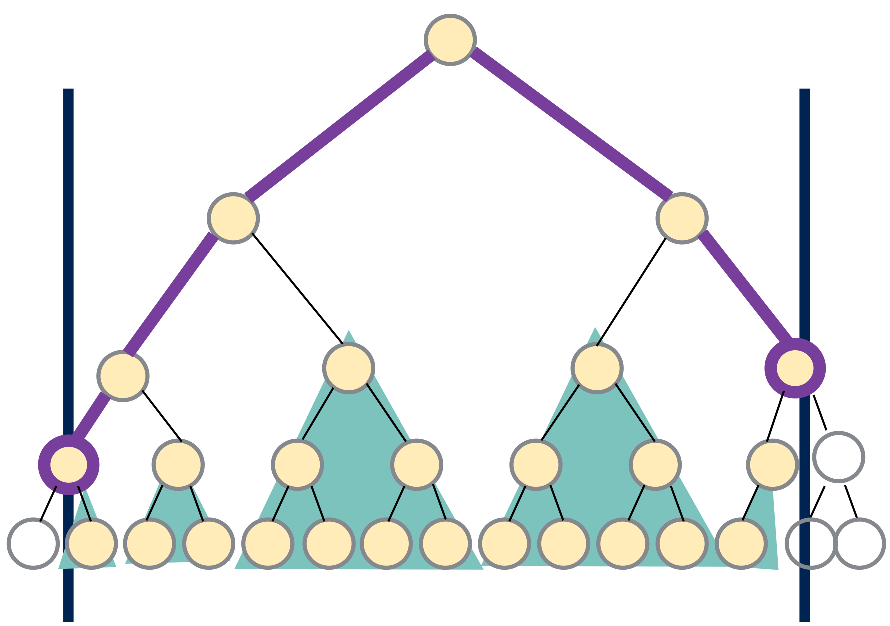
A closer look

- Use BBST to find all points with the x-coordinates in [x1, x2]

  The points in $[x_1, x_2]$ sit in $O(\lg n)$ subtrees

- Of all these points, we need those with the y-coordinates in $[y_1, y_2]$

  For each subtree we need all points in $[y_1, y_2]$



For each of these subtrees
we'll build data structure for
searching on y-coord

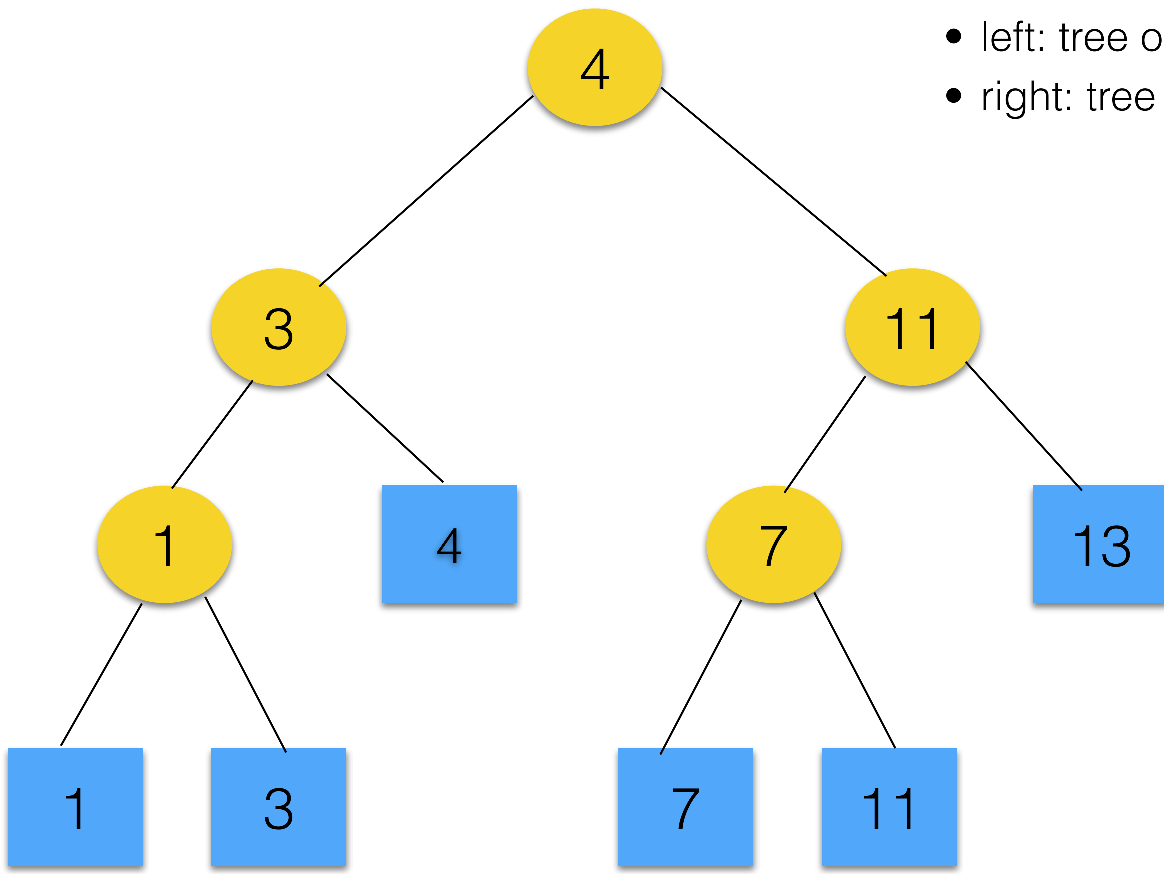
What is a good data structure for
searching on y?
A BBST by y-coord!

BBST

- We'll use a variant of BBSTs that store all data in leaves (it makes details simpler)

Example: $P = \{ 1, 3, 4, 7, 11, 13 \}$

- root: median of P
- left: tree of first half
- right: tree of second half



Class work

- Show the BBST with all data in leaves for P = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

- Write pseudocode for the algorithm to build BBST(P)
  - root: median of P
  - left: tree of first half
  - right: tree of second half

```
//P: a set of keys
//create BBST of P with all keys in leaves and return its root
buildBBST ( P )
```

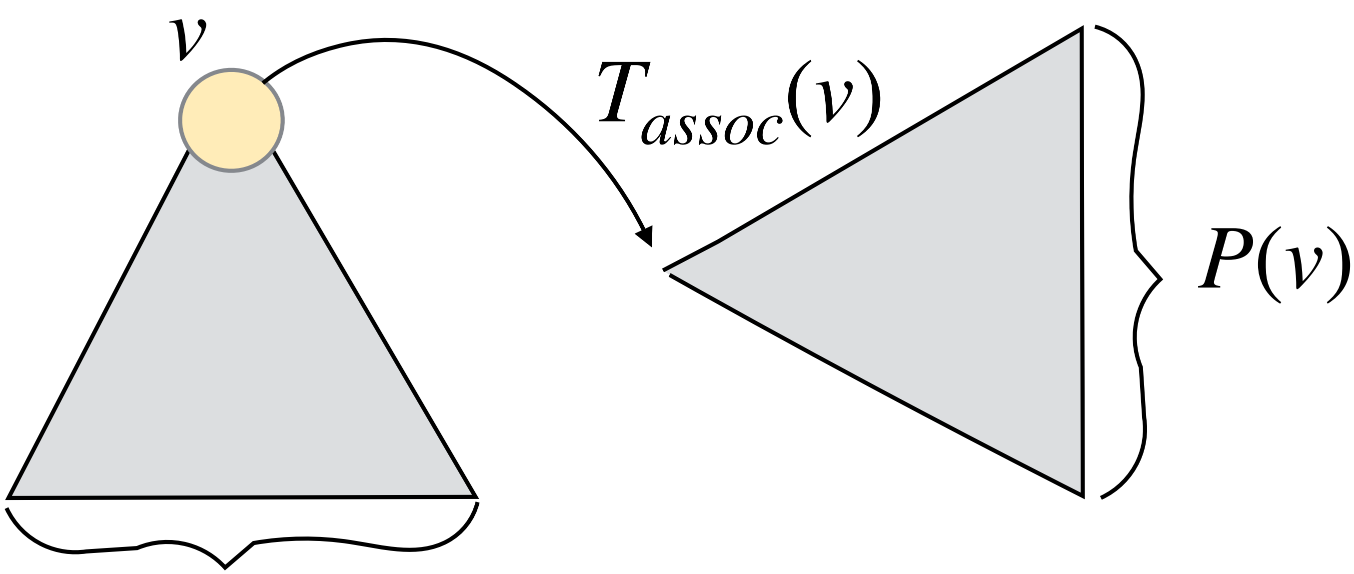
- Running time analysis in general.
- Running time analysis if P is given sorted.

And finally, the 2D Range Tree!

P: set of points

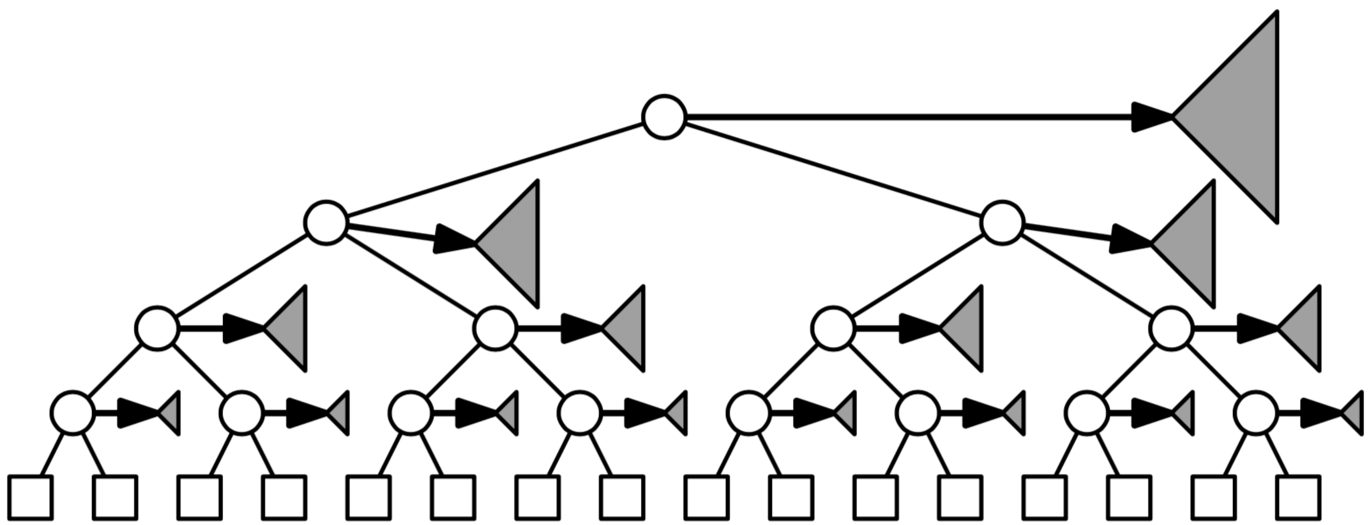
RangeTree(P) is

- A BBST T of P ordered by x-coord
- Each node v in T stores an associated structure T_{assoc} that's a BBST of $P(v)$ ordered by y-coord

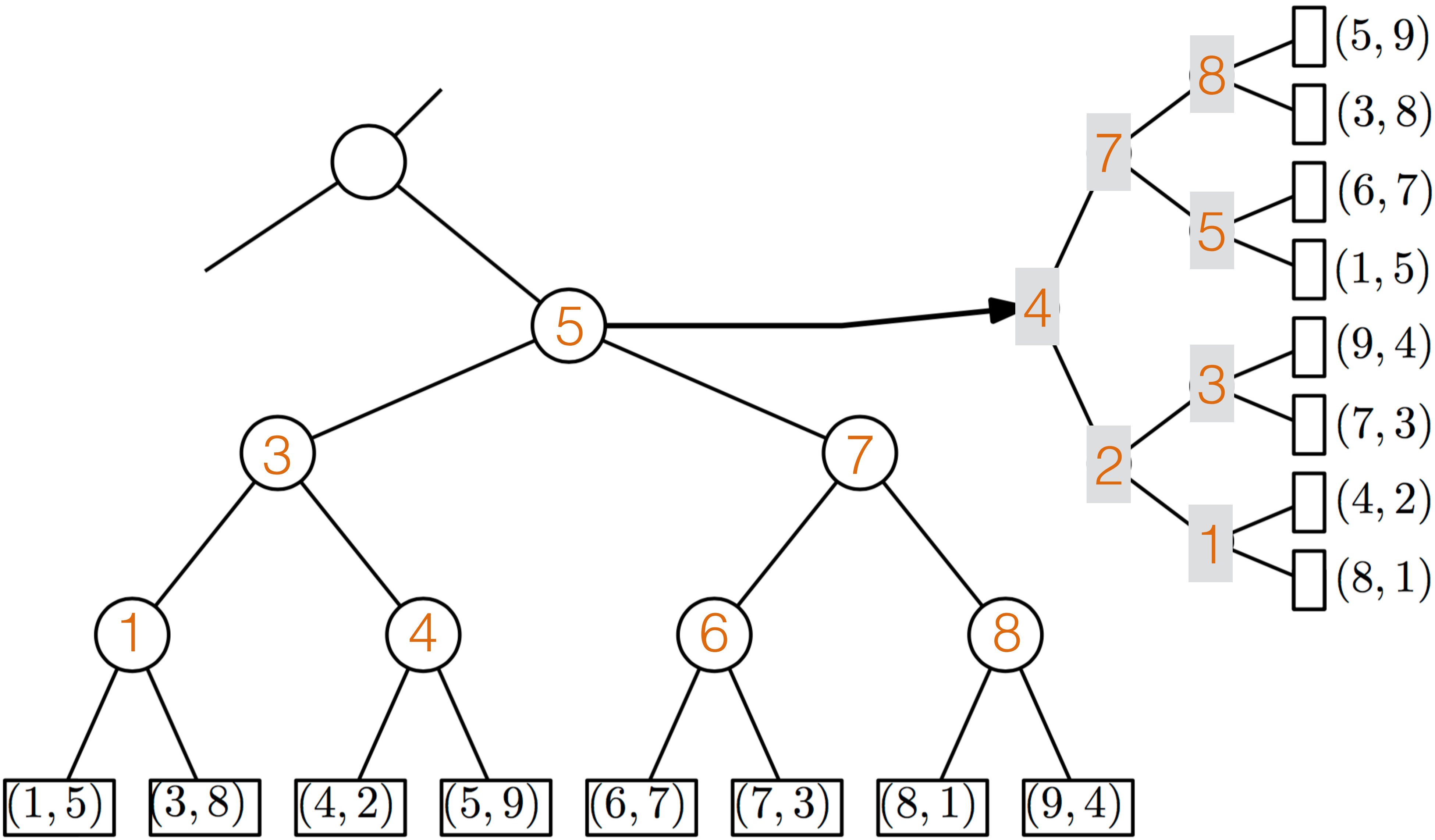


$P(v)$: all points in subtree rooted at v

Every internal node stores a whole tree in an *associated structure*, on y-coordinate



screen shot from Mark van Kreveld slides, http://www.cs.uu.nl/docs/vakken/ga/slides5b.pdf)



screen shot from Mark van Kreveld, http://www.cs.uu.nl/docs/vakken/ga/slides5b.pdf)

The 2D Range Tree

Questions

- How to build one and how fast?
- How much space does it use?
- How to answer range queries and how fast?

Building a 2D Range Tree

// param: P is an array of points
// return: builds the 2d-range tree for P and returns its root

Build2DRT(P)

1. if P contains only one point:
   - create a leaf v storing this point, (create its T_{assoc} which is just leaf) and return v
2. else
   - Construct the associated structure:
     - $root_{assoc}$ = buildBBST(P) // ordered by the y-coordinates
   - Partition P into 2 sets w.r.t. the median coordinate x_{middle}:
     - P_{left} = {p in P with p_x <= x_{middle}}, P_{right}= {p in P with p_y > x_{middle}}
   - $vleft$ = **Build2DRT(Pleft)**
   - $vright$ = **Build2DRT(P_{right})**
   - Create a node v
     - $v.x = x_{middle}$
     - $v.left = vleft$
     - $v.right = vright$
     - $v.assoc = root\_assoc$
   - return v

Building a 2D Range Tree

Running time:

- Let $T(n)$ be the time of **Build2DRT(P)** , on a set P of n points
- $T(n)$ consists of:
  - building the associated structure, a BBST on n keys: $O(n \lg n)$ (if P not sorted by y)
  - finding the median and partition: $O(n)$
  - two recursive calls
- Overall $T(n) = 2T(n/2) + O(n \lg n)$
  - This solves to $T(n) = O(n \lg^2 n)$

- Remember our trick?

Building a 2D Range Tree

- Common trick: pre-sort P and pass it as argument

```
//Px is set of points sorted by x-coord
//Py is set of points sorted by y-coord
Build2DRT(Px, Py)
```

- Maintain the sorted sets through recursion

P_{left}-sorted-by-x, P_{left}-sorted-by-y,

P_{right}-sorted-by-x, P_{right}-sorted-by-y

- If the keys are in order, a BBST can be built in $O(n)$
- We have $T(n) = 2T(n/2) + O(n)$ which solves to $O(n \lg n)$

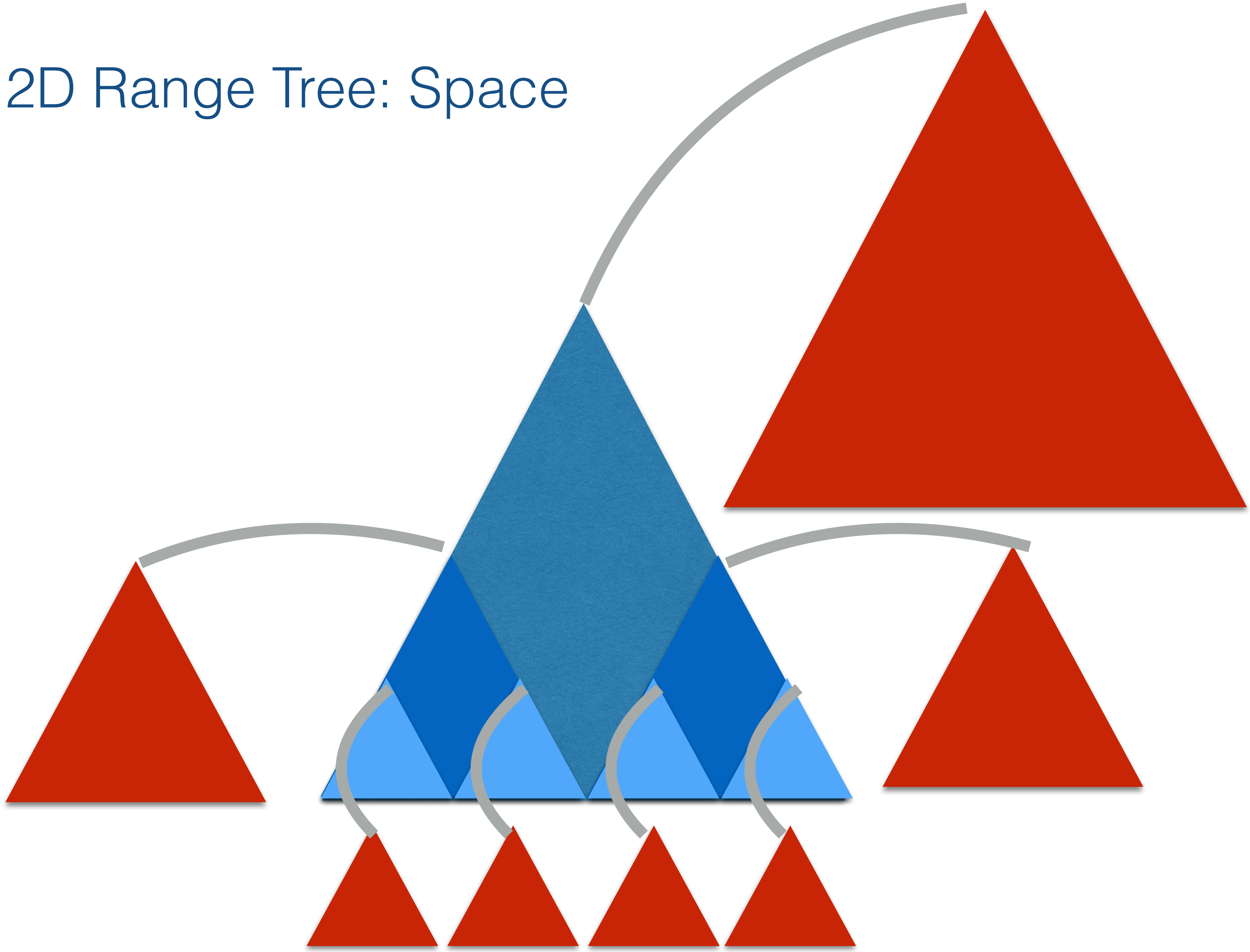
Theorem: A 2d-range tree for a set of n points can be built in $\Theta(n \lg n)$ time.

Class work

Show the range tree for

$$p_1 = (1,4), p_2 = (5,8), p_3 = (4,1), p_4 = (7,3), p_5 = (3,2), p_6 = (2,6), p_7 = (8,7)$$

The 2D Range Tree: Space



The 2D Range Tree: Space

Theorem: A 2d-range tree for a set of n points in the plane has $\Theta(n \lg n)$ size.

Two arguments:

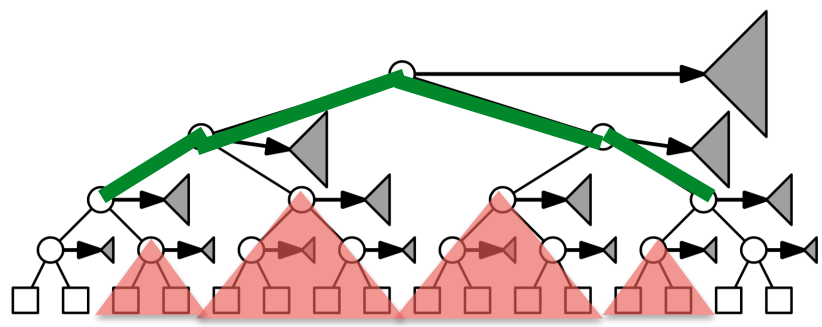
- At each level in the tree, each point is stored exactly once (in the associated structure of precisely one node). So every level stores all points and uses O(n) space => O(n lg n)

  Or: Each point p is stored in the associated structures of all nodes on the path from root to p. So one point is stored O(lg n) times => O(n lg n)

Range queries with the 2D Range Tree

- Find the split node x_{split} where the search paths for x_1 and x_2 split
- Follow path root to x_1: for each node v to the **right** of the path, query its associated structure $T_{assoc}(v)$ with $[y_1, y_2]$
- Follow path root to x_2: for each node v to the **left** of the path, query its associated structure $T_{assoc}(v)$ with $[y_1, y_2]$
- (Also, while traversing these paths, check and report the nodes on the path)

Every internal node stores a whole tree in an *associated structure*, on y-coordinate

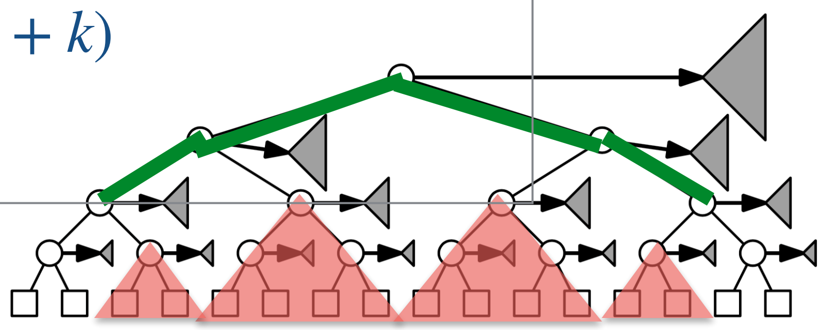


How long does this take?

Range queries with the 2D Range Tree

- There are $O(\lg n)$ subtrees in between the paths
- We query each one of them using its associated structure
- Querying T_{assoc} takes $O(\lg n_v + k')$
- Overall it takes $\sum O(\lg n_v + k') = O(\lg^2 n + k)$

Every internal node stores a whole tree in an *associated structure*, on y-coordinate



n_v: number of points in T_{assoc}

k': number of points in T_{assoc} that are in [y1,y2]

Theorem: A 2d-range tree for a set of n points answers range queries in $O(\lg^2 n + k)$ time.

We like logs!! Also, it is known how to improve this to $O(\lg n + k)$ time.

1D

- Balanced BinarySearchTree
  - Build: O(n lg n)
  - Space: O(n)
  - Range queries: O(lg n +k)

2D

- kd-trees
  - Build: O(n lg n)
  - Space: O(n)
  - Range queries: $O(\sqrt{n} + k)$

- Range trees
  - Build: O(n lg n)
  - Space: O(n lg n)
  - Range queries: O(lg n + k)

Different trade-offs

Kd-tree vs Range Tree: Does it really matter?

in 2D

n	$\log n$	$\log^2 n$	$\sqrt{n}$
16	4	16	4
64	6	36	8
256	8	64	16
1024	10	100	32
4096	12	144	64
16384	14	196	128
65536	16	256	256
1M	20	400	1K
16M	24	576	4K

YES!

screen shot from Mark van Kreveld slides, http://www.cs.uu.nl/docs/vakken/ga/slides5b.pdf)

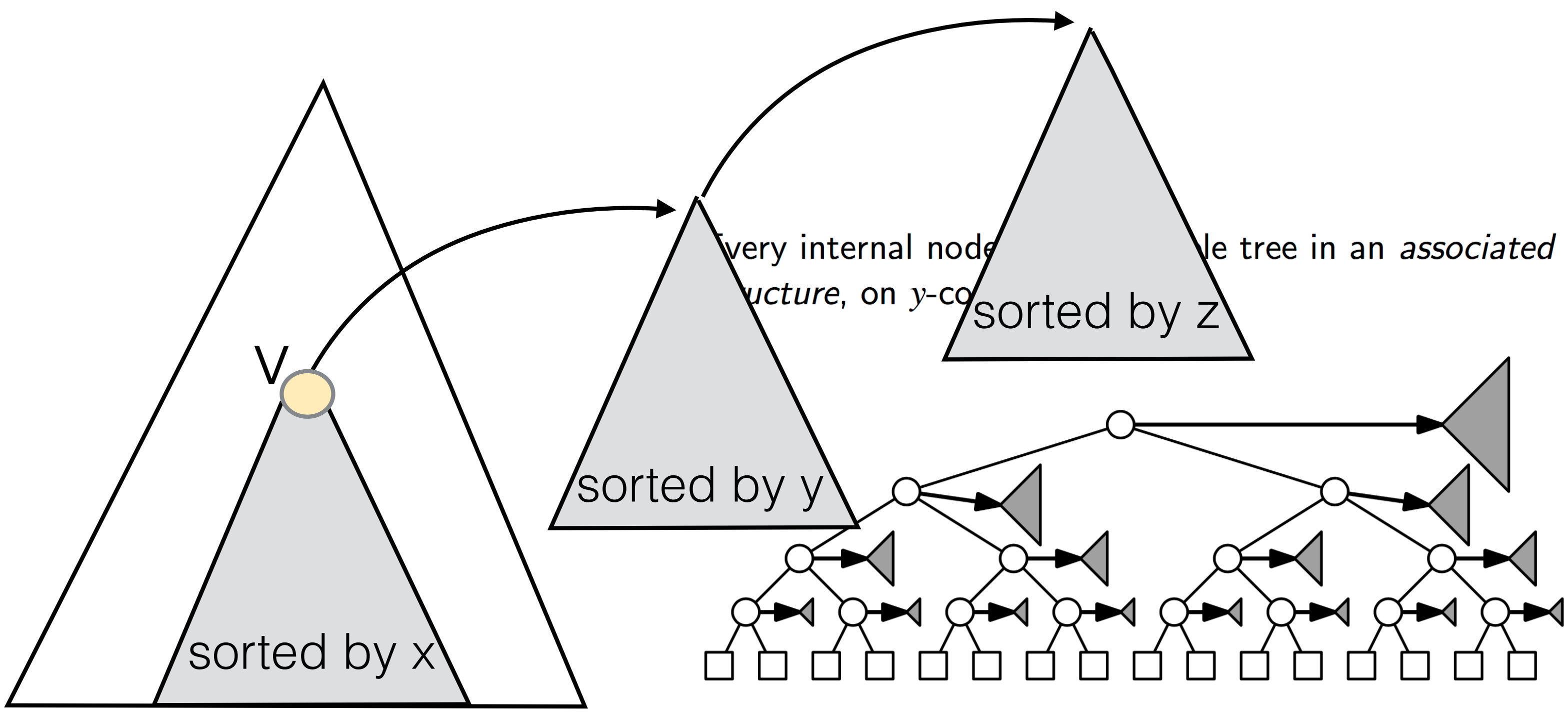
Range-trees generalize easily to d-dimensions

3D Range Trees

P: set of points in 3D

3DRangeTree(P) is

- A BBST T of P ordered by x-coord
- Each node v in T stores an associated structure T_{assoc} that's a 2D range tree for $P(v)$



3D Range Trees

Build time: $O(n \lg^2 n)$

- Think recursively
- Let $B_3(n)$ be the time to build a 3D Range Tree of n points
- Find a recurrence for $B_3(n)$
  - Think about how we build it : we build an associated structure for P that's a 2D range tree; then we build recursively a 3D range tree for the left and right half of the points
  - $B_3(n) = 2B_3(n/2) + B_2(n)$
  - This solves to $O(n \lg^2 n)$

3D Range Trees

Size: $O(n \lg^2 n)$

- Why? we can thinks of this in two ways:
- An associated structure for n points uses $O(n \lg n)$ space. Each point is stored in all associated structures of all its ancestors =>$O(n \lg^2 n)$
- Or, recursively
    - Let $S_3(n)$ be the size of a 3D Range Tree of n points
    - Find a recurrence for $S_3(n)$
    - We build an associated structure for P that's a 2D range tree; then we build recursively a 3D range tree for the left and right half of the points
    - $S_3(n) = 2S_3(n/2) + S_2(n)$
    - This solves to $O(n \lg^2 n)$

3D Range Trees

Query:

- Query BBST on x-coord to find $O(\lg n)$ nodes (roots of subtrees)
- Then perform a 2D range query in each node

Time:

- Let $Q_3(n)$ be the time to answer a 3D range query
- Find a recurrence for $Q_3(n)$
  - $Q_3(n) = O \lg n) + O(\lg n) \times Q_2(n)$
  - This solves to $O(n \lg^3 n + k)$

Kd-tree vs Range Tree

4D

n	$\log n$	$\log^4 n$	$n^{3/4}$
1024	10	10,000	181
65,536	16	65,536	4096
1M	20	160,000	32,768
1G	30	810,000	5,931,641
1T	40	2,560,000	1G

screen shot from Mark van Kreveld slides, http://www.cs.uu.nl/docs/vakken/ga/slides5b.pdf)

Class work

Show the 3D-range tree for the the set of points below: