Geometric primitives

Computational Geometry [csci 3250]
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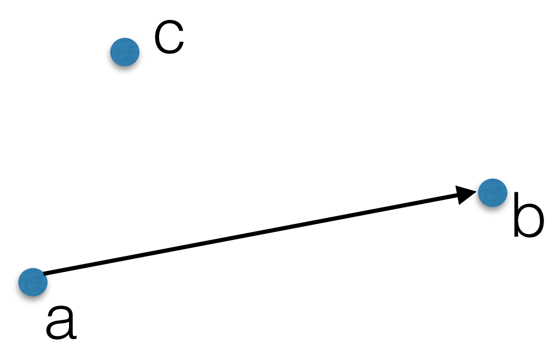
Given 3 points in the plane, we want to answer the following question:



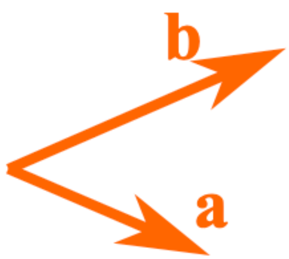
Is c left of line ab?

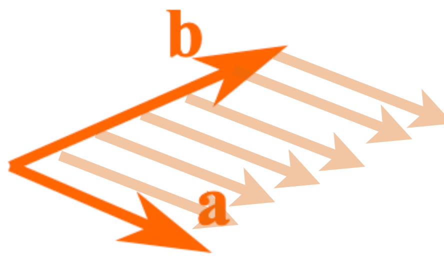
```
//return true if c is (strictly) left of ab, false otherwise

bool left(point2d a, b, c)
```

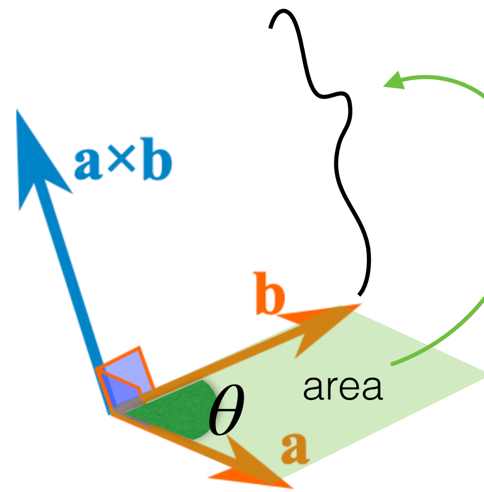
- Why? This will be our basic primitive and based on it we'll develop others (e.g. do two segments ab and cd intersect? is a point inside a polygon? etc)
- To answer, we'll use the sign of the cross product

The cross product in 3D

- Given two vectors a, b in 3D



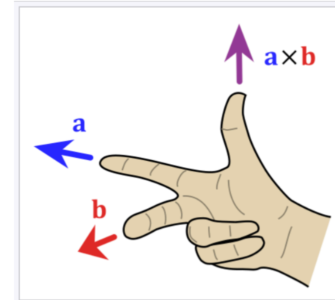


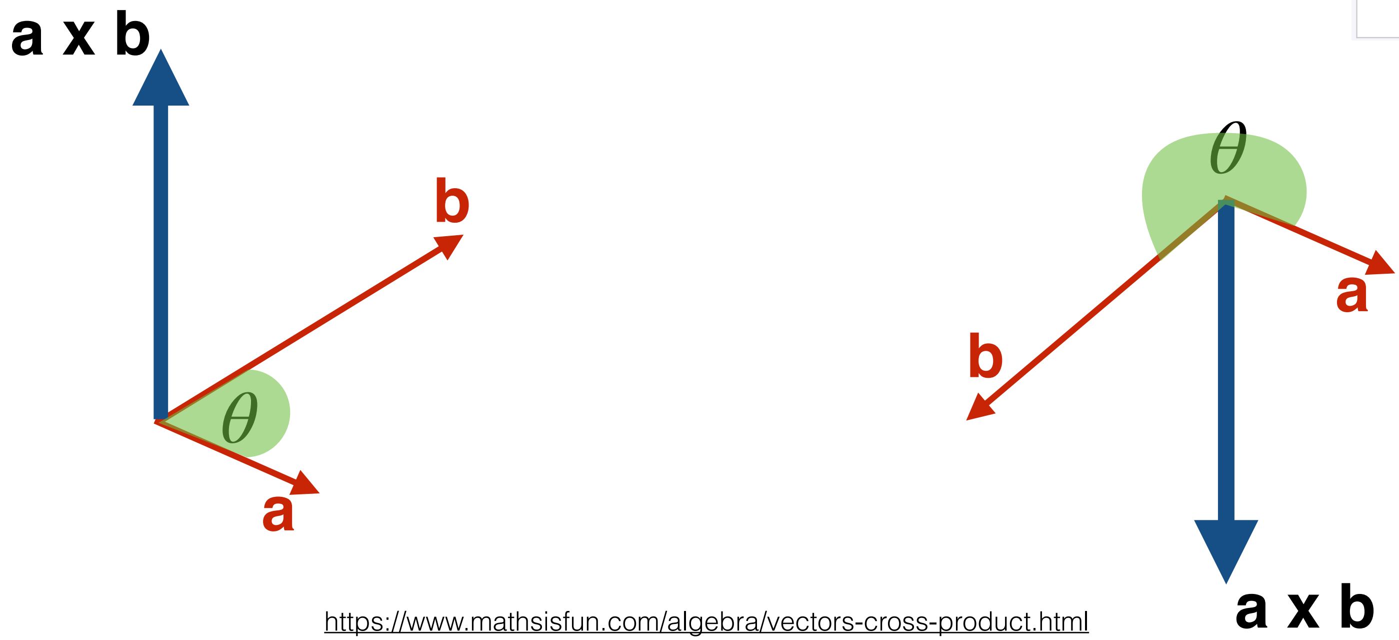


- The cross product $a \times b$ is a vector
  - direction: perpendicular to a and b
  - its magnitude: the area of the parallelogram defined by a and b: $|a| \cdot |b| \cdot sin(\theta)$

There are two possible directions perpendicular on a, b.

The direction of the cross product: the right hand rule





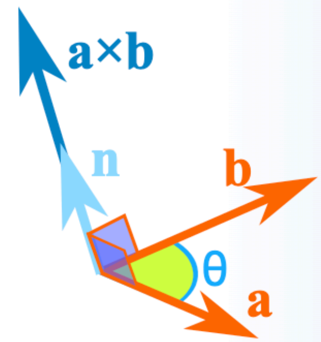
https://www.mathsisfun.com/algebra/vectors-cross-product.html

If θ is the angle from a to b, note that

- $\sin(\theta) > 0$ when $\theta < \pi$ (b is left of a)
- $\sin(\theta) < 0$ when $\theta \in (\pi, 2\pi)$ (b is right of a)
- $\sin(\theta) = 0$ when $\theta = 0$ or $\theta = \pi$ (a, b on same line)

If we knew the sign of $\sin(\theta)$ we could tell if a is left/right of b

The cross product in 3D



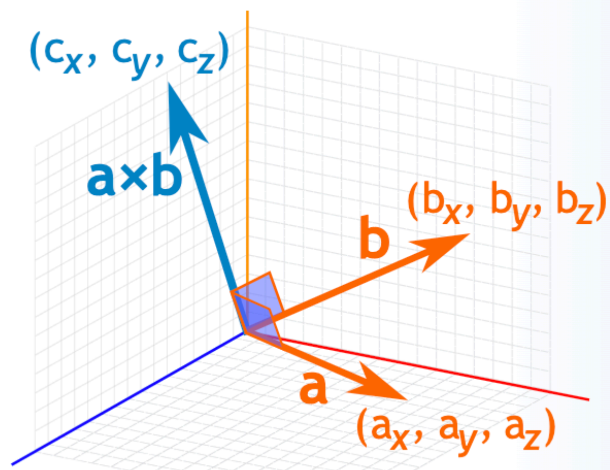
We can calculate the cross product as:

$$a \times b = \vec{n} \cdot \text{(area parallelogram)} = \vec{n} \cdot |a||b|\sin(\theta)$$

where $\vec{n}$ is the unit vector in the direction of the normal

Also as:

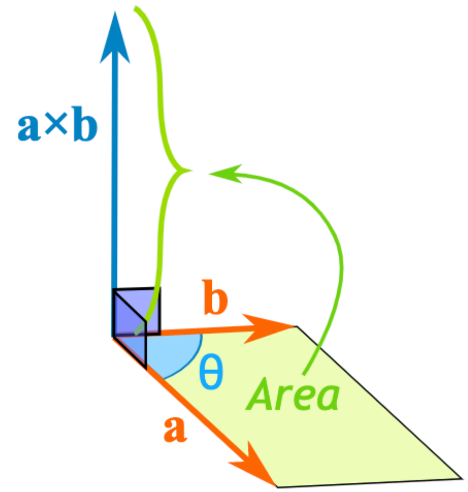
$$a \times b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a.x & a.y. & a.z \\ b.x & b.y & b.z \end{vmatrix} = \vec{i} \begin{vmatrix} a.y & a.z \\ b.y & b.z \end{vmatrix} - \vec{j} \begin{vmatrix} a.x & a.z \\ b.x & b.z \end{vmatrix} + \vec{k} \begin{vmatrix} a.x & a.y \\ b.x & b.y \end{vmatrix}$$



pics from: https://www.mathsisfun.com/algebra/vectors-cross-product.html

If the vectors are in 2D:

$$a = \begin{bmatrix} a.x \\ a.y \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} b.x \\ b.y \\ 0 \end{bmatrix}$$



$$a \times b = \vec{k} \cdot \text{ (area parallelogram) } = \vec{k} \cdot |a||b|\sin(\theta)$$

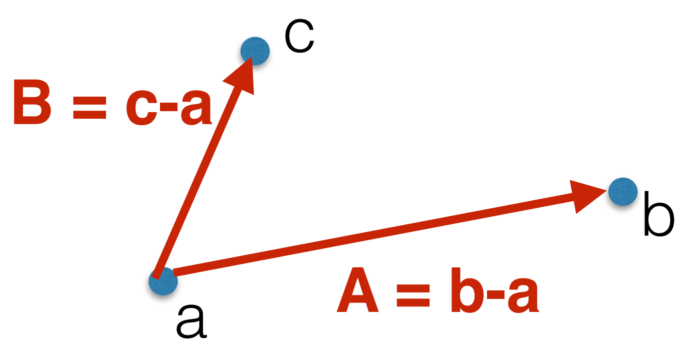
$$a \times b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a.x & a.y. & 0 \\ b.x & b.y & 0 \end{vmatrix} = \vec{k} \begin{vmatrix} a.x & a.y \\ b.x & b.y \end{vmatrix} = \vec{k}(a.x \cdot b.y - a.y \cdot b.x)$$

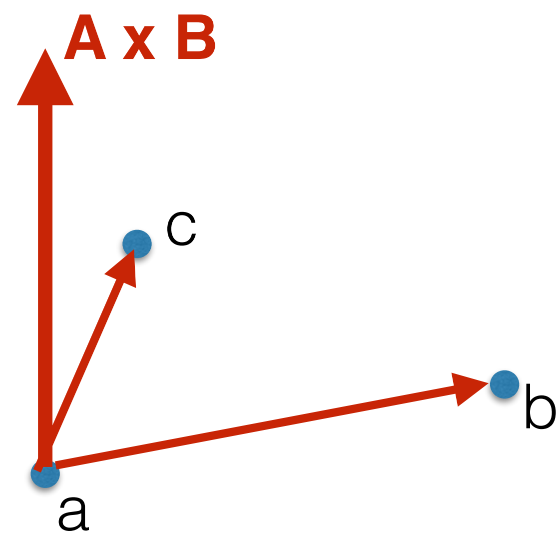
this is a signed quantity

- if $> 0 \Rightarrow \sin(\theta) > 0$ => b left of a
- if $< 0 \Rightarrow \sin(\theta) < 0$ => b right of a
- if $= 0 \Rightarrow$ a,b on same line

pics from: https://www.mathsisfun.com/algebra/vectors-cross-product.html

is point c left of line ab?





$$A = \begin{bmatrix} b.x - a.x \\ b.y - a.y \end{bmatrix}$$

$$B = \begin{bmatrix} c.x - a.x \\ c.y - a.y \end{bmatrix}$$

$$A \times B = \vec{k}(A.x \cdot B.y - A.y \cdot B.x)$$

this is a signed quantity $= 2 \cdot signedArea(abc)$

- if $> 0 \Rightarrow$ B=ac left of A=ab => c left of ab
- if $< 0 \Rightarrow$ B=ac right of A=ab => c right of ab
- if $= 0 \Rightarrow$ a, b, c collinear

```
//return 2 x the area of the triangle from ab to c
int two_signed_area(point2d a, b, c) {
   Ax = b.x - a.x;
   Ay = b.y - a.y;
   Bx = c.x -a.x;
   By = c.y - aa.y;
   return Ax By - Ay Bx;
  }
```

```
//return true if c is (strictly) left of ab, false otherwise
bool left(point2d a, b, c) {
   return two_signed_area(a, b, c) > 0;
}
```

```
//return true if c is (strictly) right of ab, false otherwise
bool right(point2d a, b, c) {
     return two_signed_area(a, b, c) < 0;
}
```

```
//return true if a, b, c collinear, false otherwise
bool collinear(point2d a, b, c) {
  return two_signed_area(a, b, c) == 0;
}
```