Art Gallery Problems

Computational Geometry [csci 3250]
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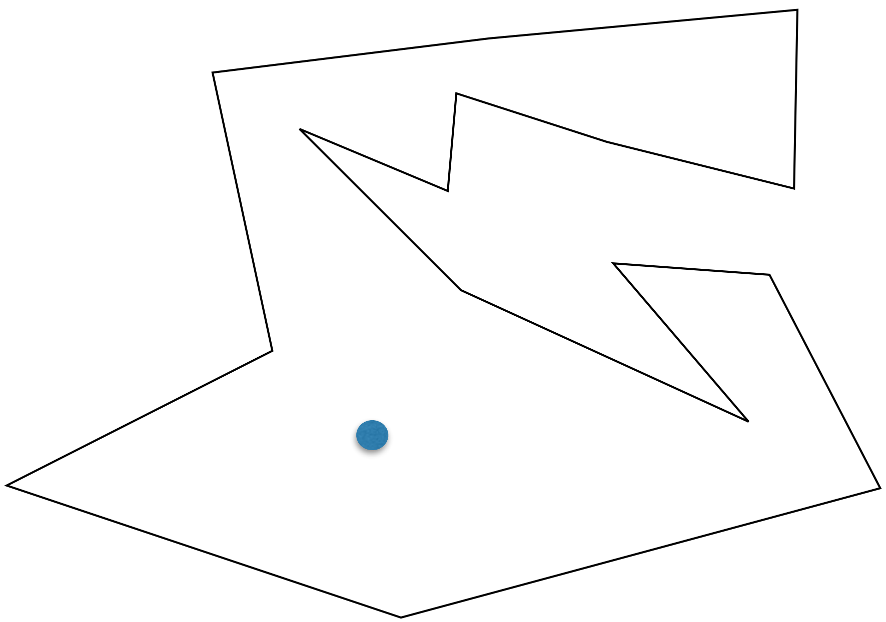
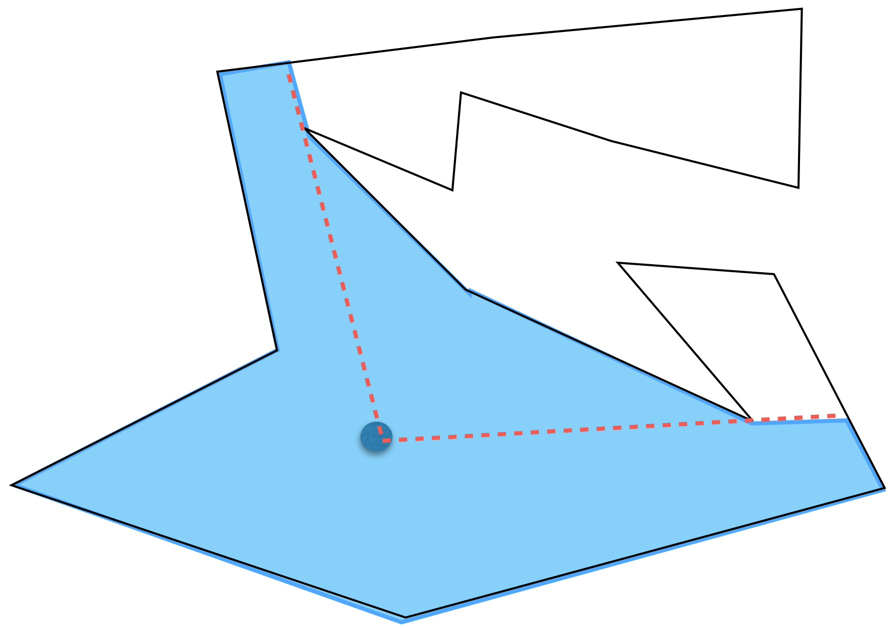
Visible polygon

Imagine an art gallery whose floor plan is a simple polygon.

Imagine a guard (a point) inside the gallery. What does the guard see?



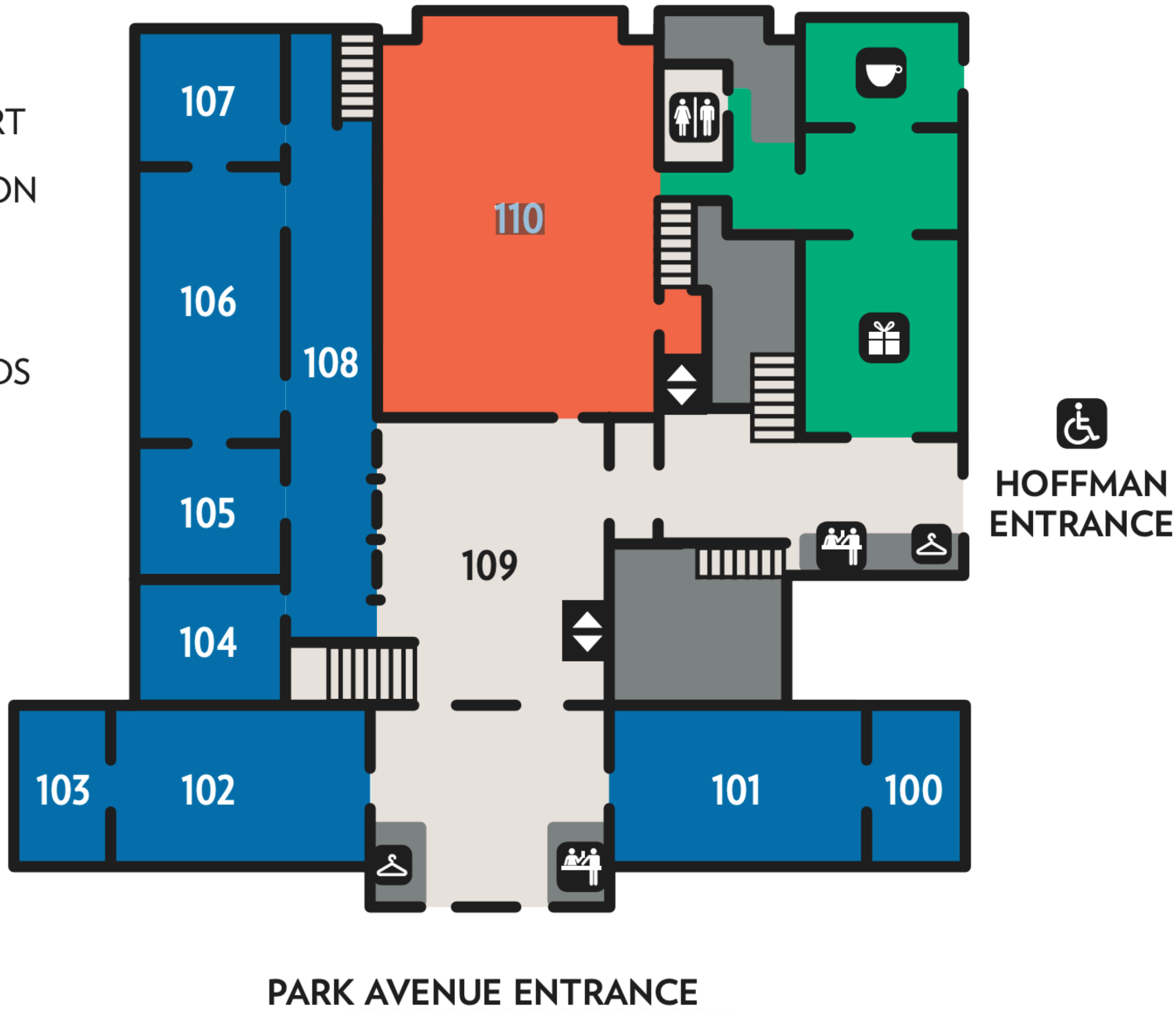


Def: two points a, b are visible if segment ab stays inside P (touching boundary is ok).

Portland OR Museum of Art floor plan

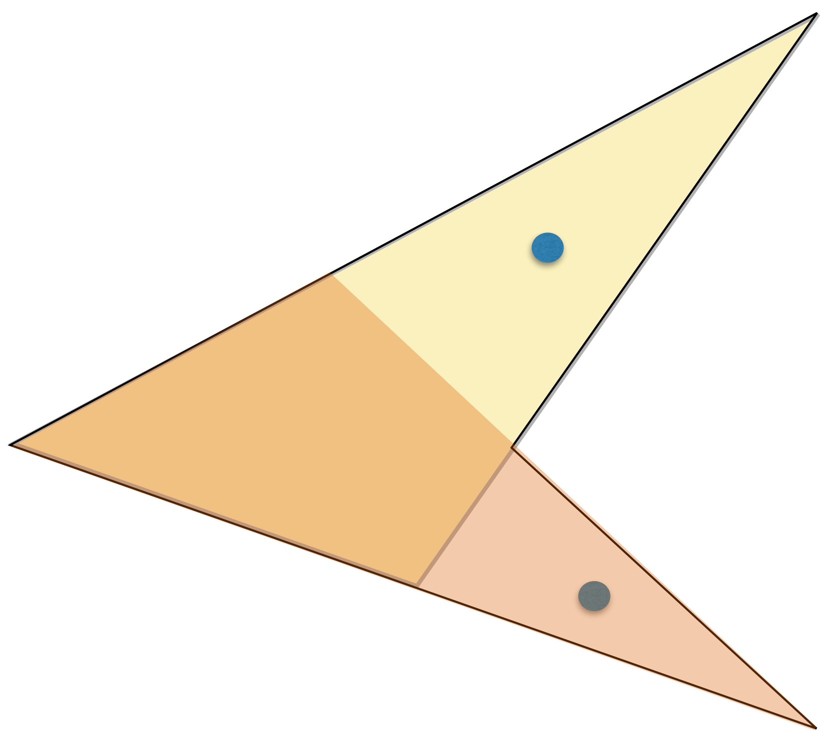
100-108	ASIAN ART
109	SCULPTURE COURT
110	SPECIAL EXHIBITION GALLERY

- MUSEUM STORE
- MUSEUM GROUNDS
- GUEST SERVICES
- RESTROOMS

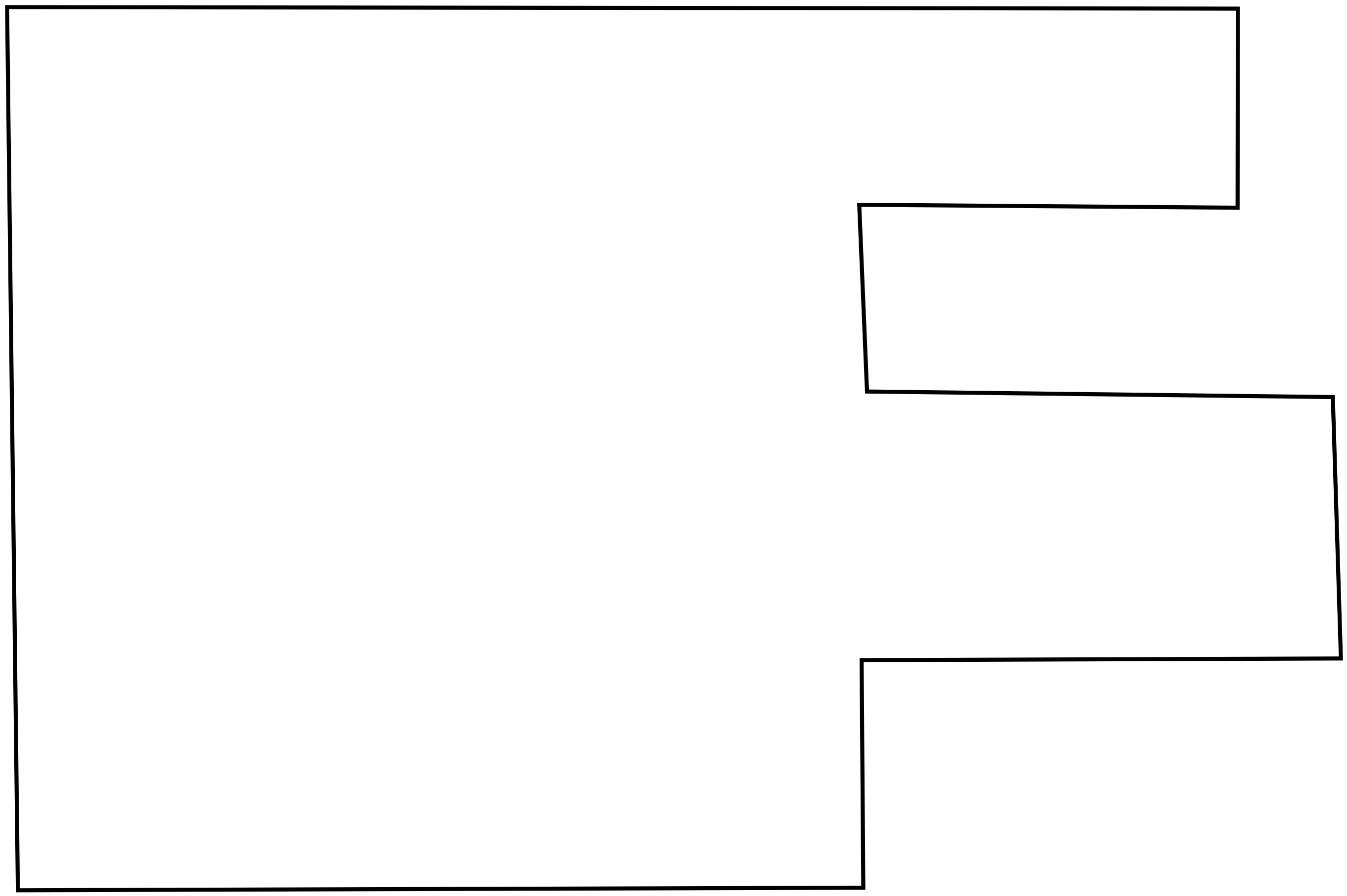


Guarding

Definition: Given a polygon P and a set of guards $G = \{g_1, g_2, ..., g_k\}$, we say that the set of guards **covers** polygon P if every point in P is visible to at least one guard.



How many guards do you need?



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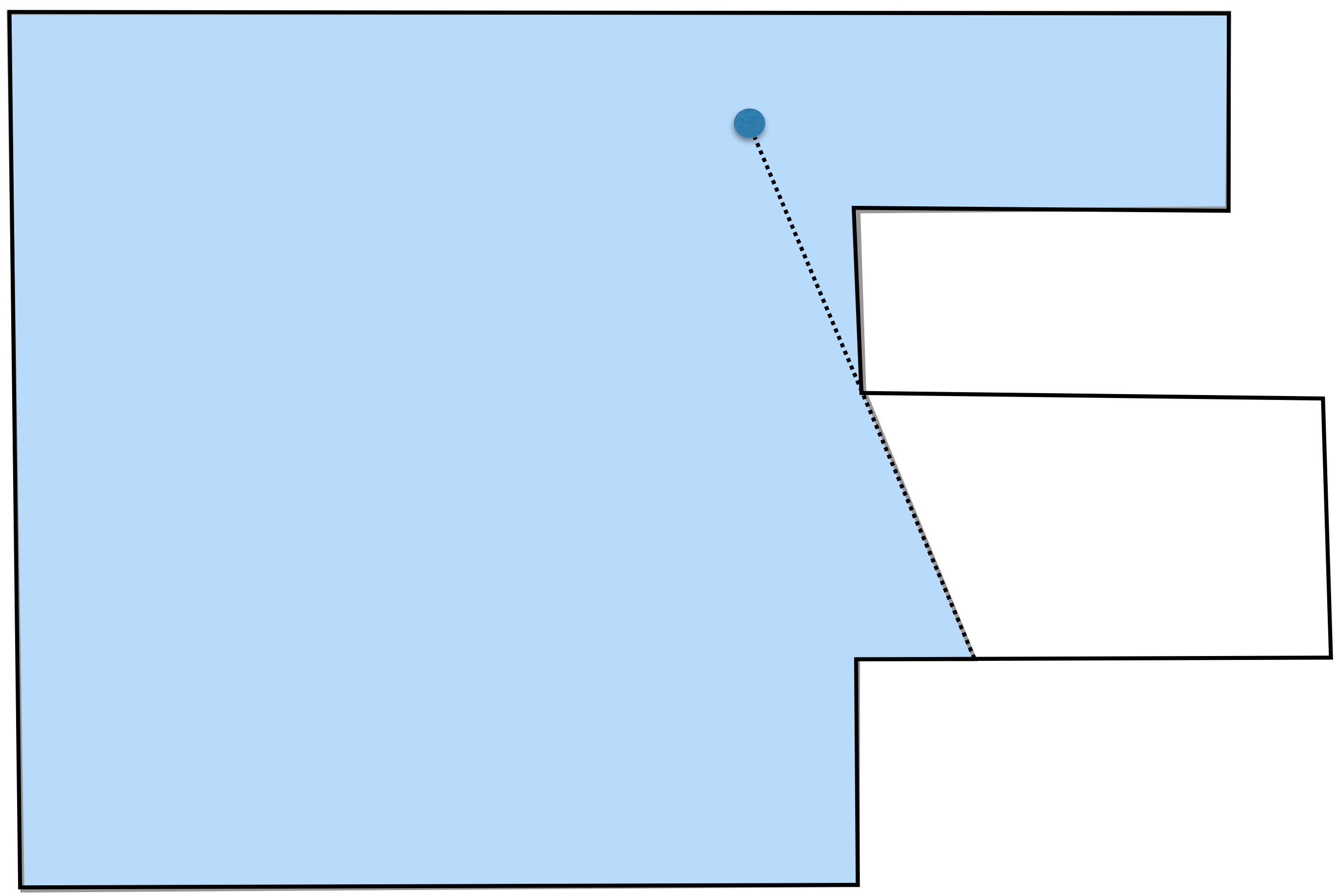

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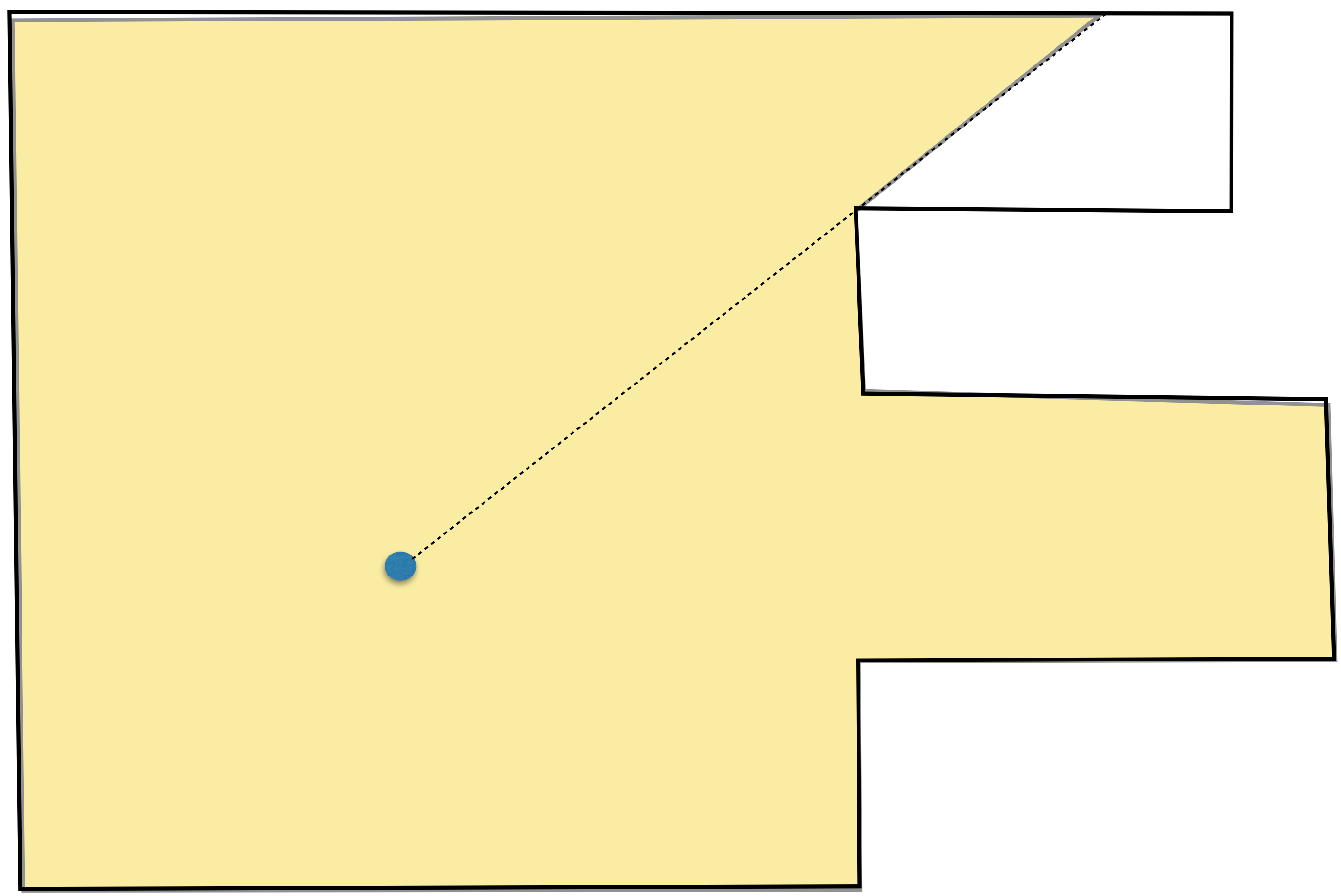
Two, and here's one way to place them



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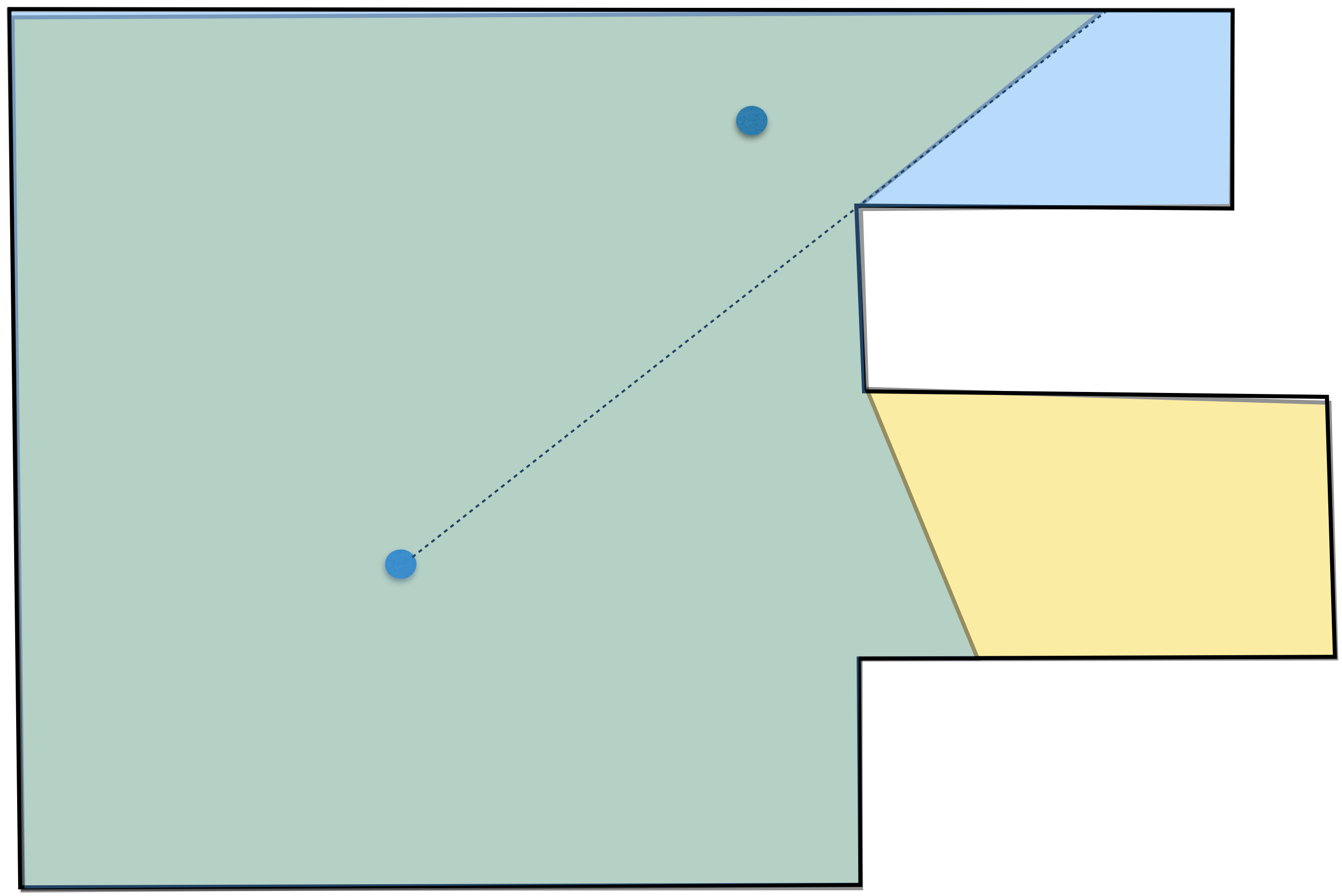
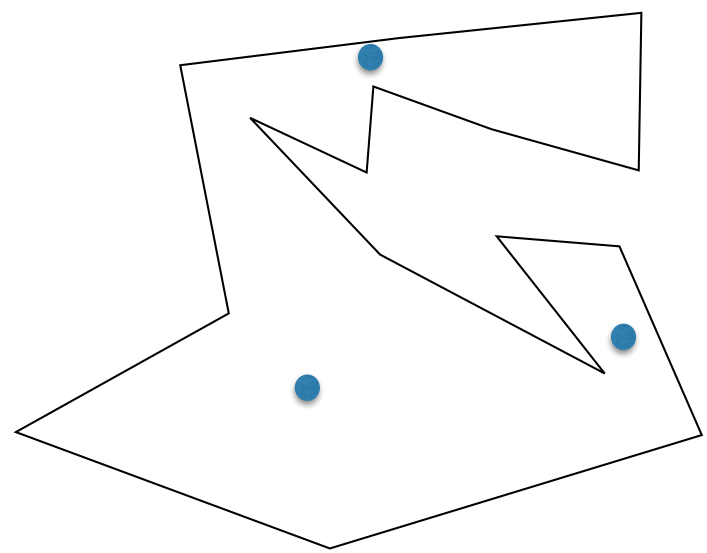


Art Gallery Problems

Here's one question we could ask: Given a polygon P of n vertices, find the smallest number of guards (and their locations) to guard P.

Notation

- Let P_n: polygon of n vertices
- Let $g(P)$ = the smallest number of guards to cover P_n
- Problem: Given P_n, find $g(P)$



It has been shown that finding $g(P)$ is NP-complete.

Art Gallery Problems

Here's another question (Klee's problem): We cannot find $g(P)$, but what is the largest $g(P)$ for all polygons of a given size?

- Consider all polygons of n vertices, and for each one, the smallest number of guards to cover it. What is the largest $g(P_n)$, for all polygons P_n?

Notation: Let $G(n) = \max\{g(P) \text{ for all } P_n\}$

Klee's problem: Find $G(n)$

Examples

G(3) = ?

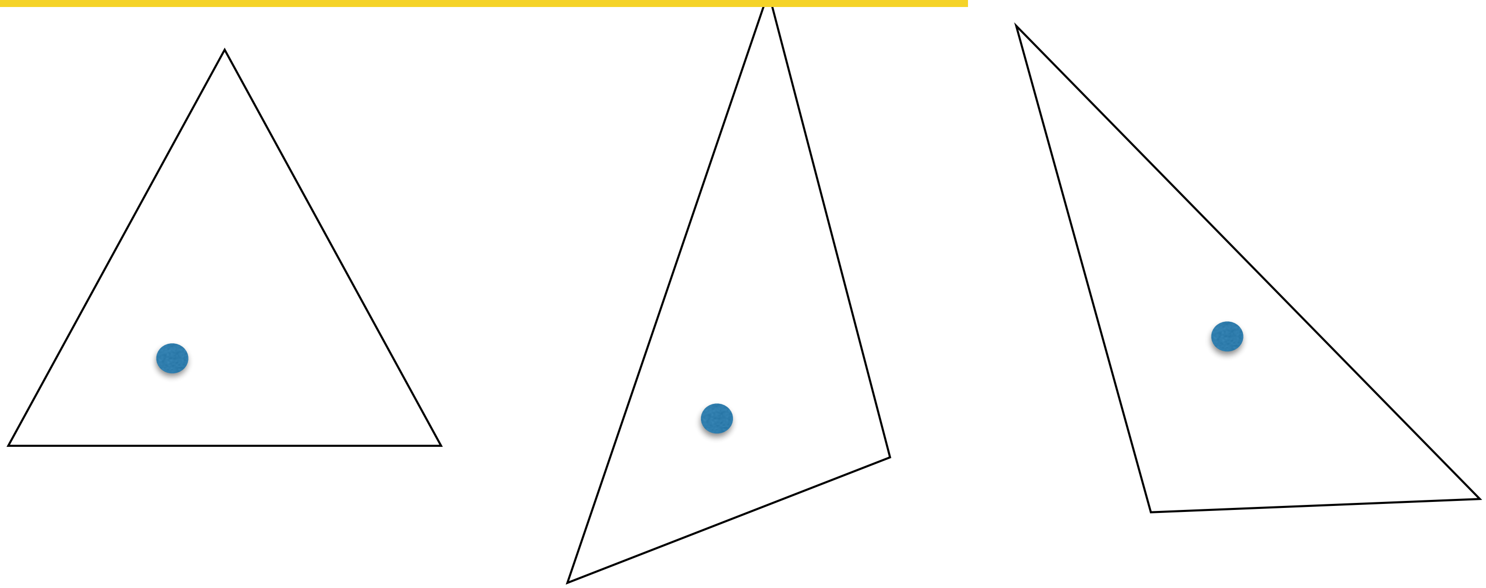
What is the smallest nb. of guards to guard a triangle?

$n = 3$

P_3 : triangle

$g(P_3)$: min nb. of guards to guard triangle P_3

$G(3)$: min nb. of guards to guard any triangle



Claim: Any triangle can be guarded with one guard. Therefore $G(3) = 1$.

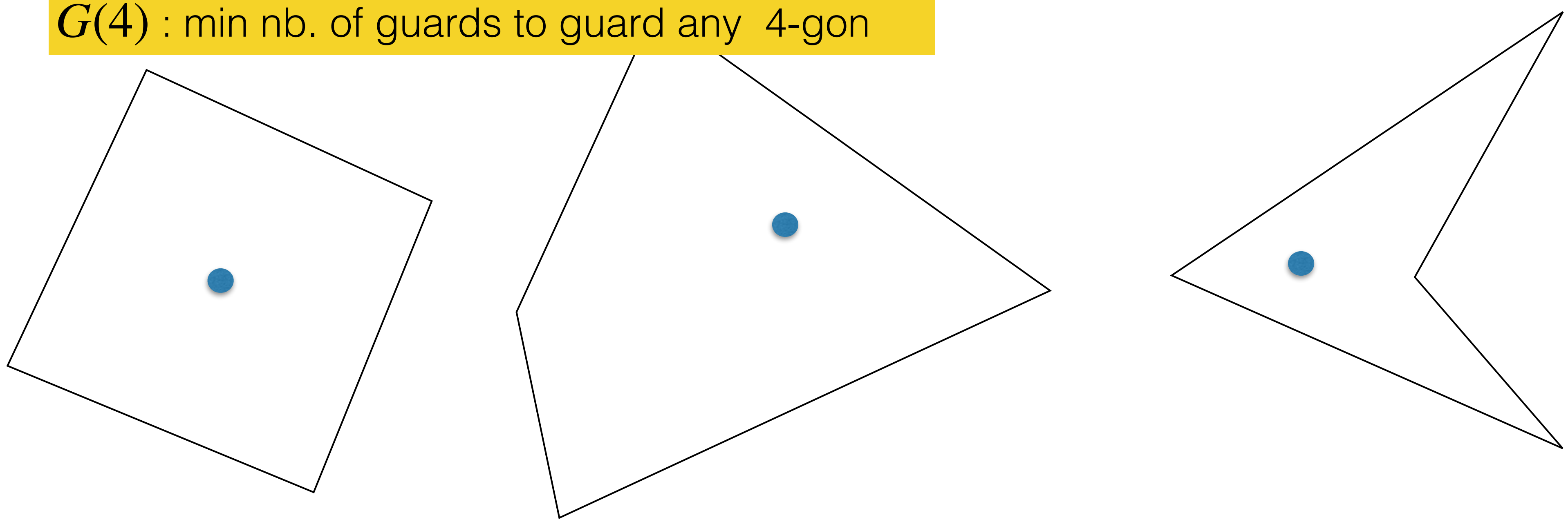
G(4) = ?

$n = 4$

P_4 : quadrilateral, or 4-gon

$g(P_4)$: min nb. of guards to guard P_4

$G(4)$: min nb. of guards to guard any 4-gon



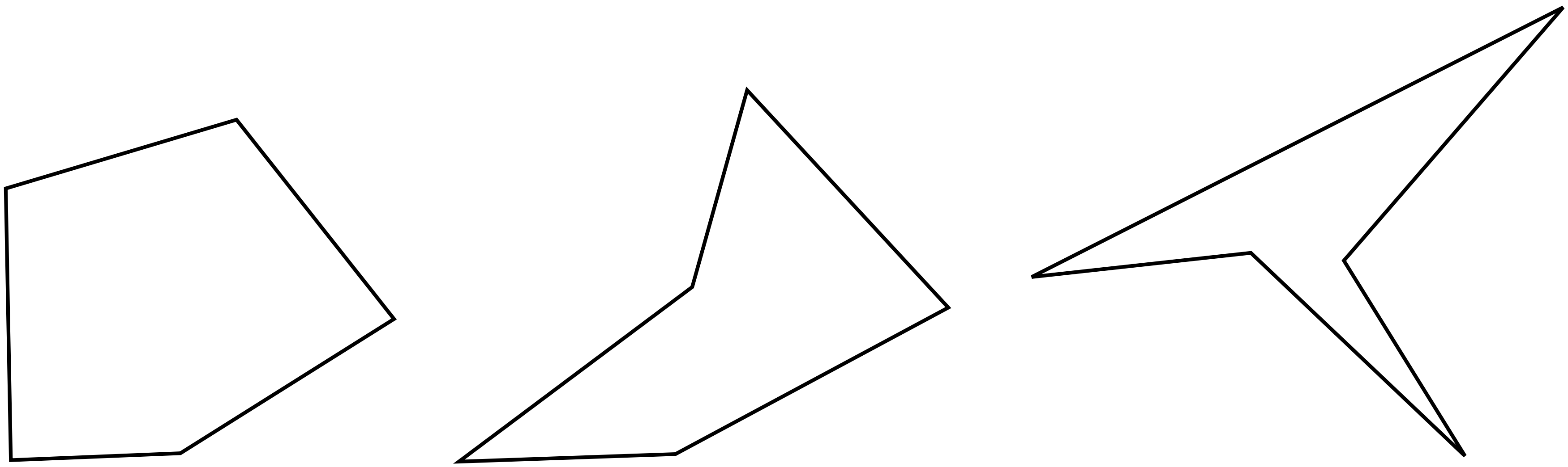
Claim: Any 4-gon can be guarded with one guard. Therefore $G(4) = 1$.

G(5) = ?

n=5, P is a 5-gon

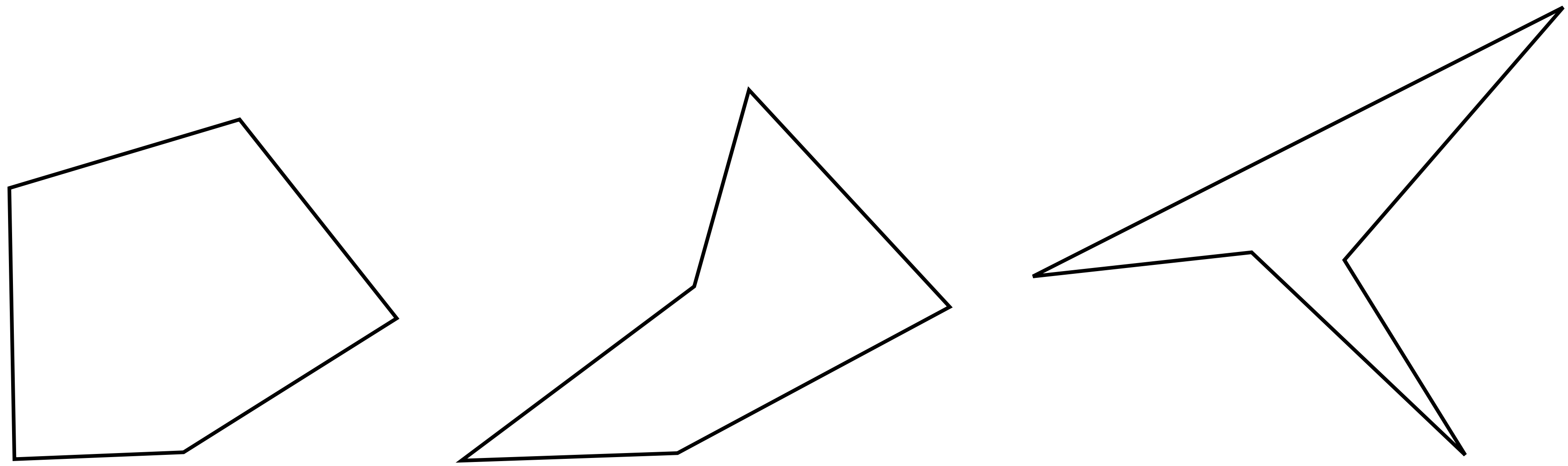
For a specific polygon P: What is $g(P)$, the smallest nb. of guards to guard P?

$G(5)$: the smallest nb. of guards to guard any 5-gon



Can any 5-gon be guarded by one point?

G(5) = ?



Can any 5-gon be guarded by one point? **YES!**

Claim: Any 5-gon can be guarded with one guard, $g(P) = 1$. Therefore $G(5) = 1$.

G(6) = ?

n=6

For a specific hexagon P: What is $g(P)$, the smallest nb. of guards to guard P?

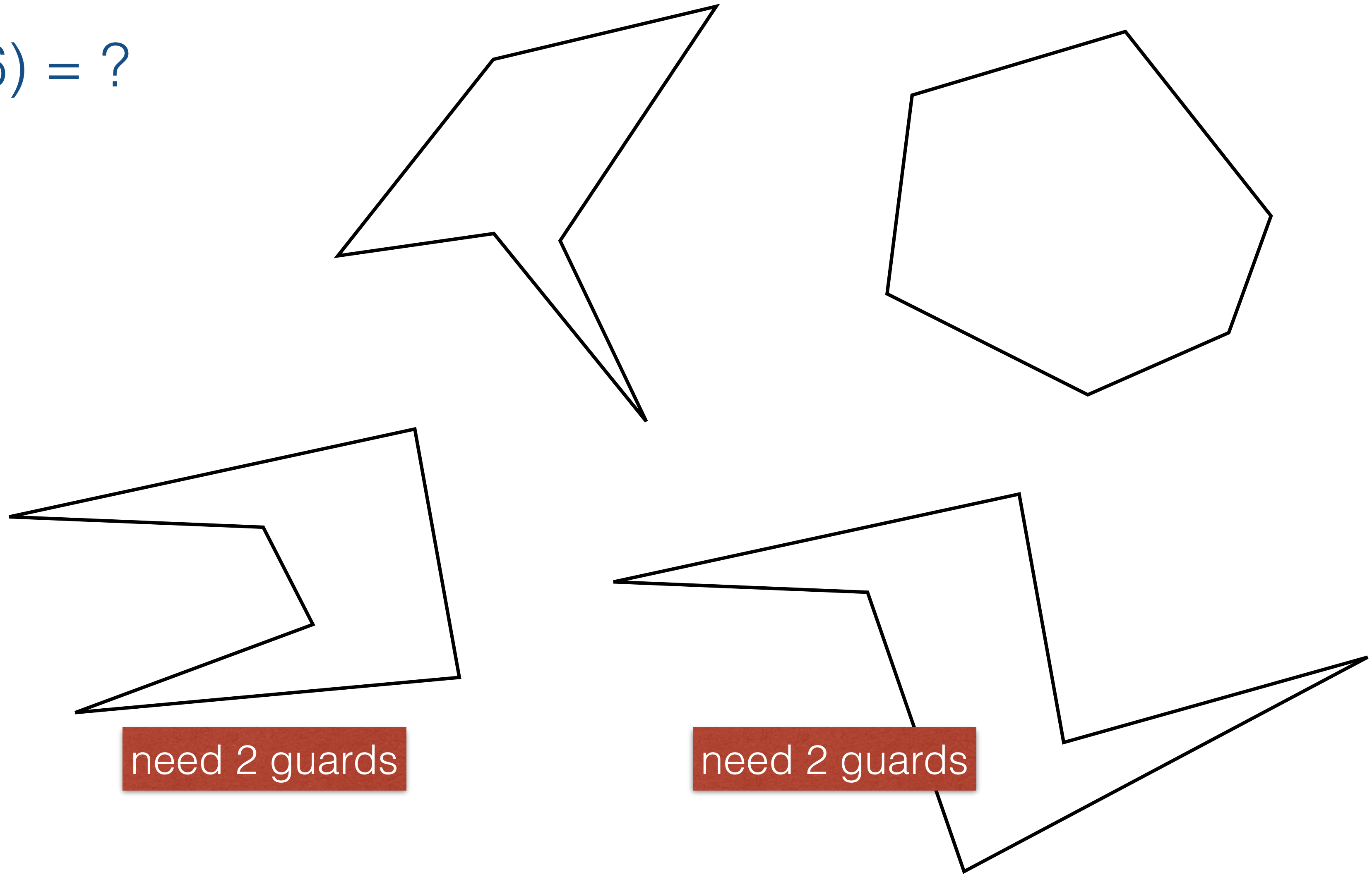
$G(6)$: the smallest nb. of guards to guard any 6-gon

Can any hexagon be guarded by one point?

Can you find a hexagon that cannot be guarded with one guard?

G(6) = ?

n=6



Claim: Any 6-gon can be guarded with at most two guards, $g(P) = 1$ or $g(P) = 2$. Therefore $G(6) = 2$.

Klee's problem: what is $G(n)$?

Our goal is to find $G(n)$ as a function of n.

Trivial bounds

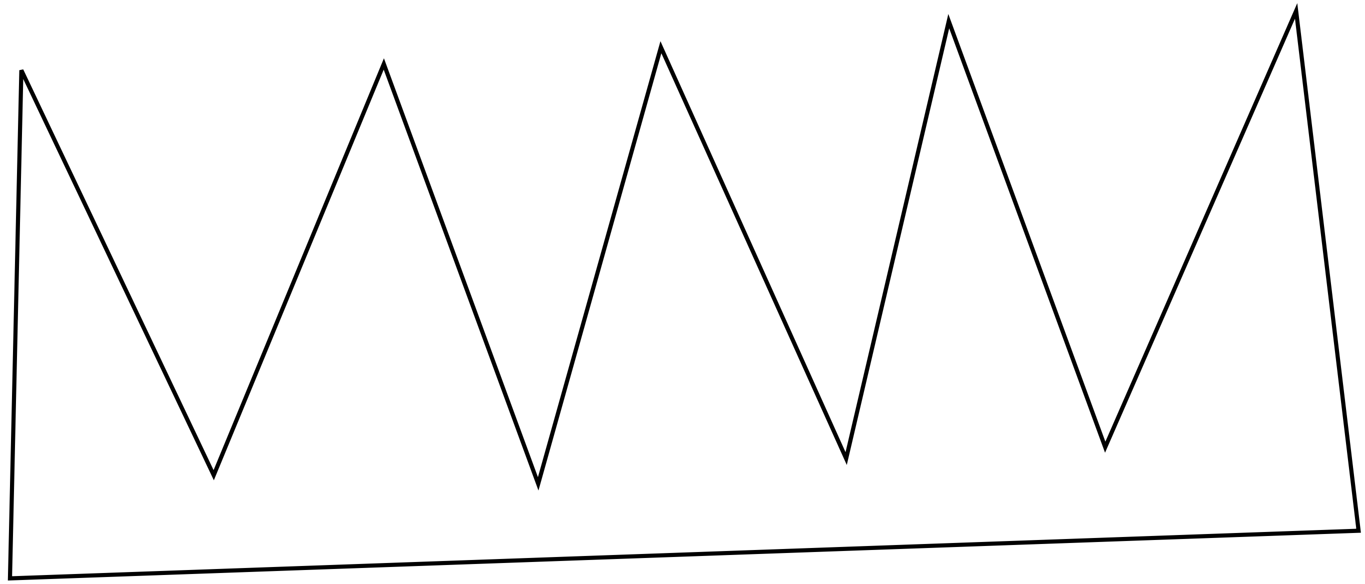
- $G(n) >= 1$: obviously, you need at least one guard.
- $G(n) <= n$: place one guard in each vertex

- $G(n)$ is the smallest number that always works for any n-gon. It is sometimes necessary and always sufficient to guard a polygon of n vertices.
  - $G(n)$ is necessary: there exists a P_n that requires $G(n)$ guards
  - $G(n)$ is sufficient: any P_n can be guarded with $G(n)$ guards

Klee's problem: G(n) = ?

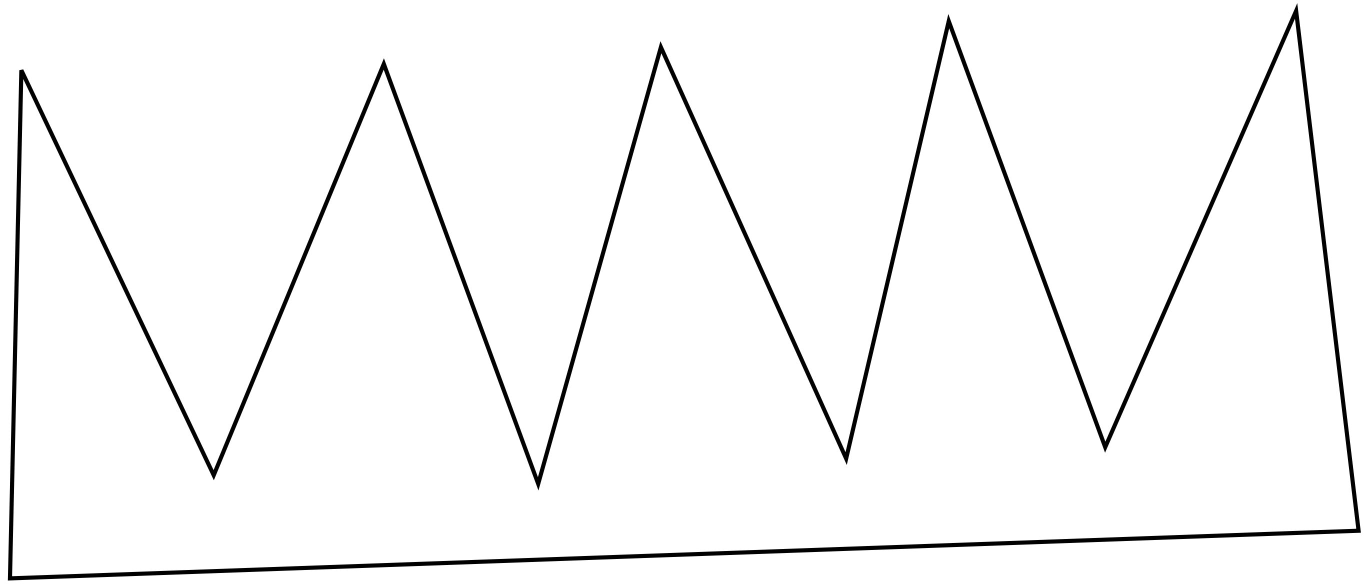
Come up with a P_n that requires as many guards as possible.

Klee's problem: G(n) = ?



How many guards does this need?

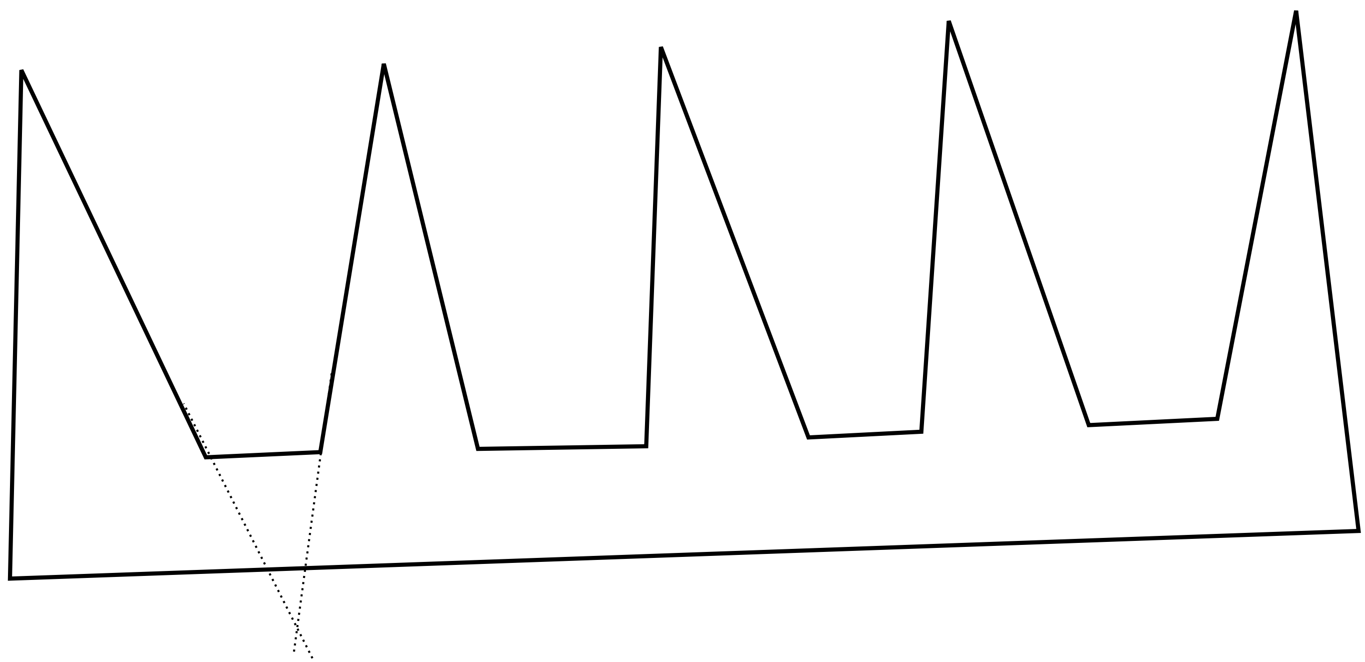
Klee's problem: G(n) = ?



How many guards does this need?

This polygon requires $\lfloor n/4 \rfloor$ guards => $G(n) \geq \lfloor n/4 \rfloor$

Klee’s problem: G(n) = ?



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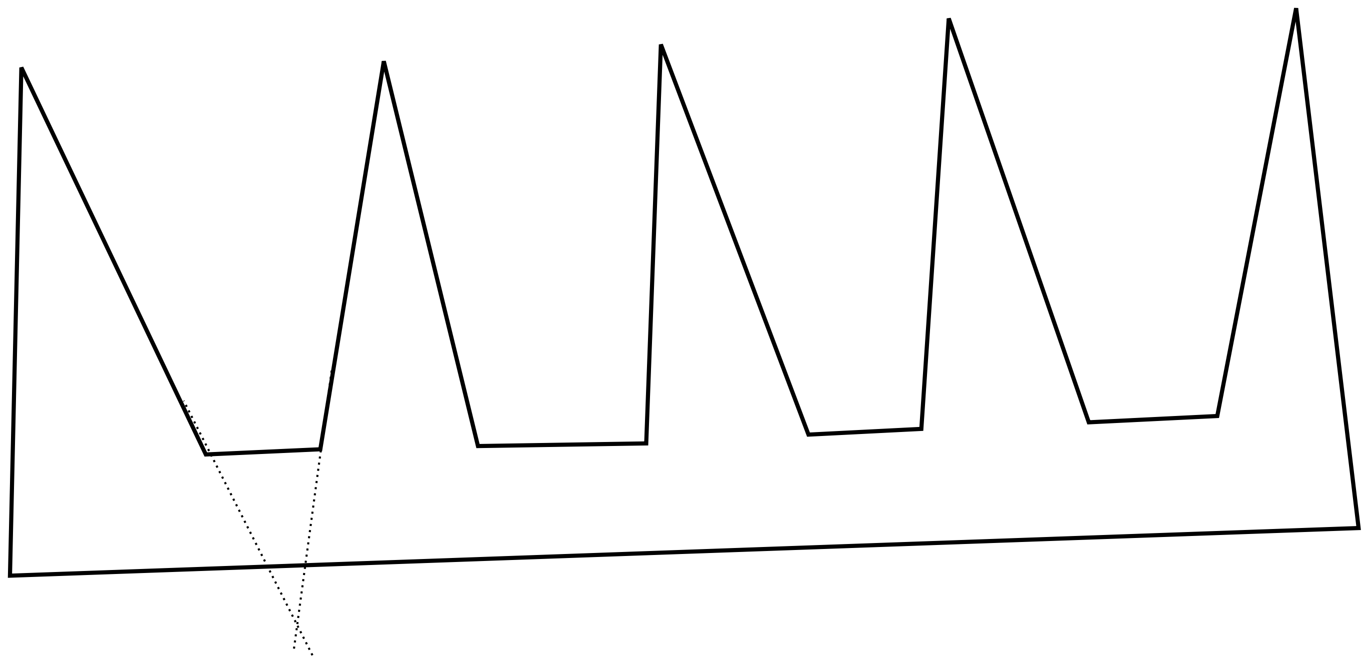


How many guards does this need?

This polygon requires $\lfloor n/3 \rfloor$ guards => $G(n) \geq \lfloor n/3 \rfloor$

Are there P_n that require more guards ?

Or, $\lfloor n/3 \rfloor$ guards always suffice for any P_n?

Klee's problem: G(n) = ?

It was shown that $\lfloor n/3 \rfloor$ is always sufficient for any P_n:

Theorem: Any P_n can be guarded with at most $\lfloor n/3 \rfloor$ guards.

- (Complex) proof by induction
- Subsequently, simple and beautiful proof due to Steve Fisk, who was Bowdoin Math faculty.
- Proof in The Book.

https://en.wikipedia.org/wiki/Proofs_from_THE_BOOK

Proofs from THE BOOK

From Wikipedia, the free encyclopedia

Proofs from THE BOOK is a book of mathematical proofs by Martin Aigner and Günter M. Ziegler. The book is dedicated to the mathematician Paul Erdős, who often referred to "The Book" in which God keeps the most elegant proof of each mathematical theorem. During a lecture in 1985, Erdős said, "You don't have to believe in God, but you should believe in The Book."

Content [edit]

Proofs from THE BOOK contains 32 sections (44 in the fifth edition), each devoted to one theorem but often containing multiple proofs and related results. It spans a broad range of mathematical fields: number theory, geometry, analysis, combinatorics and graph theory. Erdős himself made many suggestions for the book, but died before its publication. The book is illustrated by Karl Heinrich Hofmann. It has gone through five editions in English, and has been translated into Persian, French, German, Hungarian, Italian, Japanese, Chinese, Polish, Portuguese, Korean, Turkish, Russian and Spanish.

The proofs include:

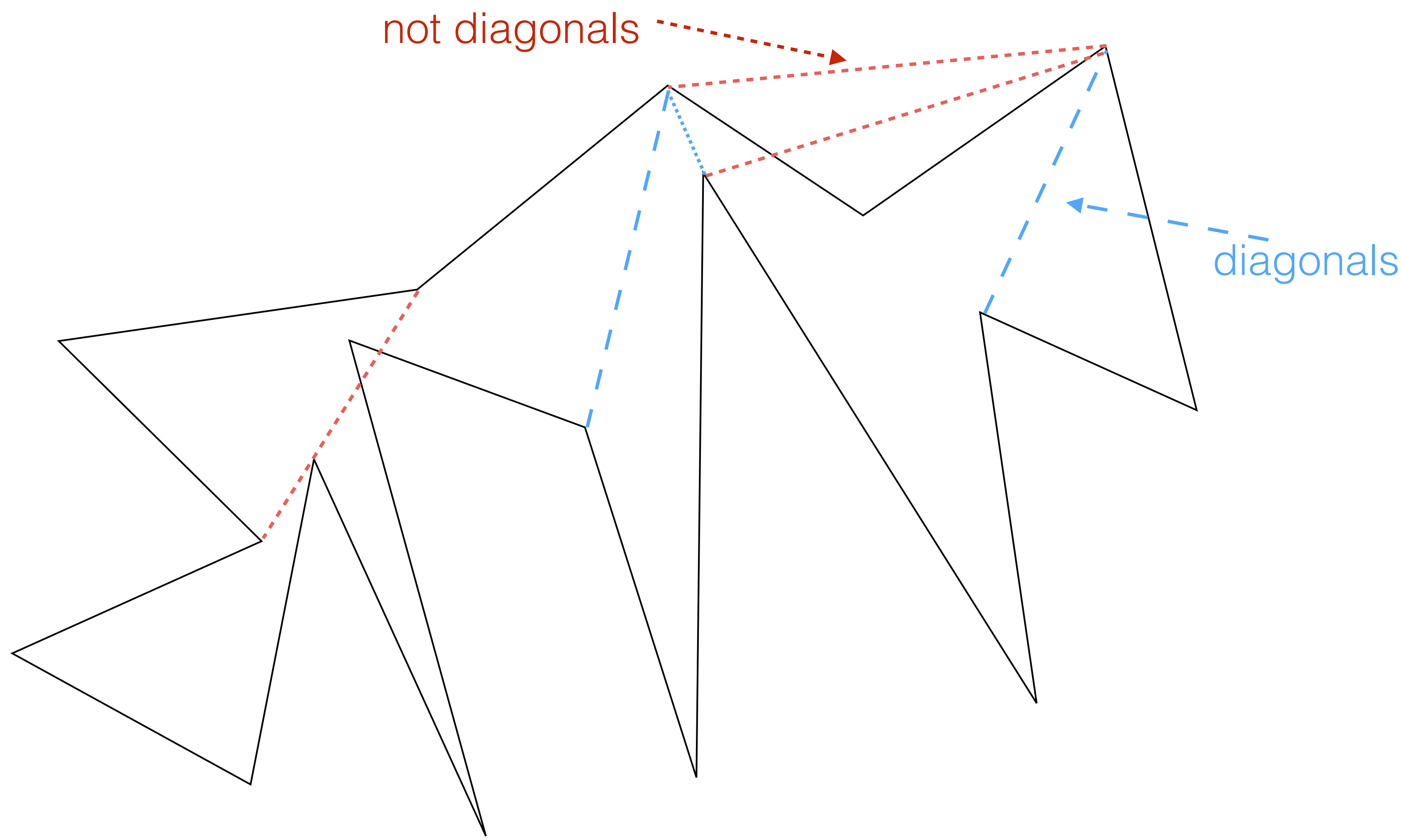
- Proof of Bertrand's postulate
- Proof that e is irrational (also showing the irrationality of certain related numbers)
- Six proofs of the infinitude of the primes, including Euclid's and Furstenberg's
- Monsky's theorem (4th edition)
- Wetzel's problem on families of analytic functions with few distinct values
- Steve Fisk's proof of the The art gallery theorem

References [edit]

Fisk's proof

First step: Polygon triangulation

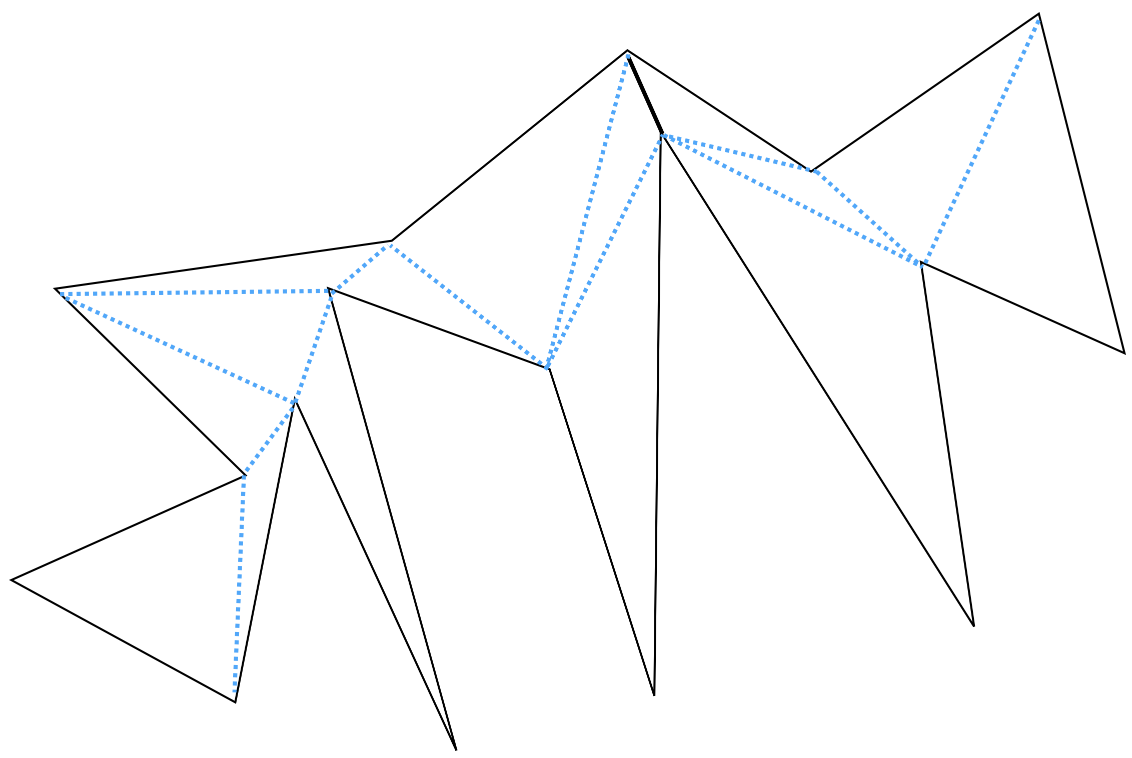
Given a simple polygon P, a **diagonal** is a segment between 2 non-adjacent vertices that lies entirely within the interior of the polygon.



Polygon triangulation

Theorem: Any simple polygon can be triangulated.

Proof: later



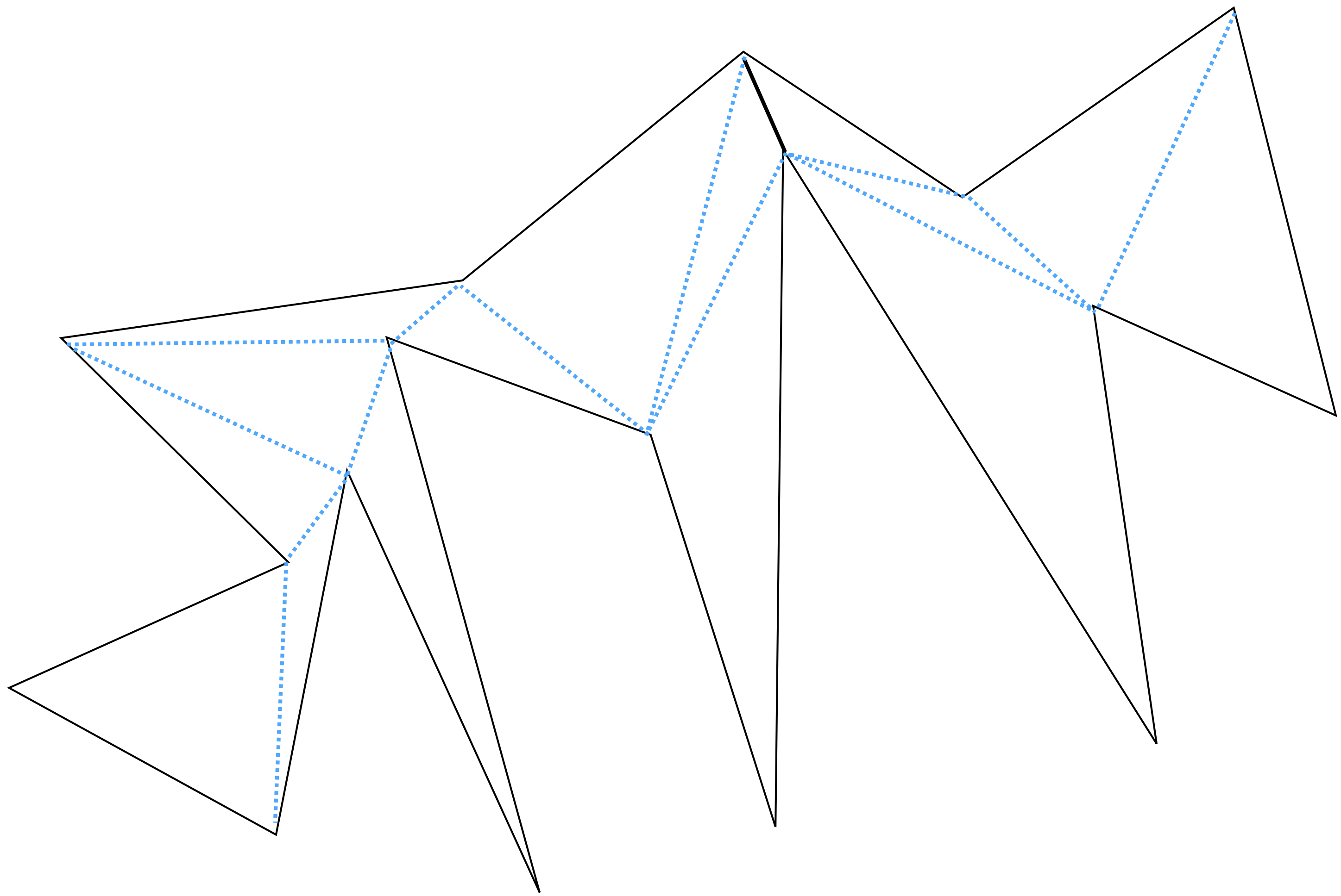
Second step: Coloring

- A coloring of a graph is an assignment of colors to vertices such that no two adjacent vertices (vertices connected by an edge) have the same color

- The chromatic number of a graph G, $\chi(G)$: the smallest nb of colors needed to color G
- Fundamental problem in graph theory
- Computing $\chi(G)$ is NP-complete
- Results:
  - Any planar graph can be 5-colored. O(n) time.
  - Any planar graph can be 4-colored (proof by computer). $O(n^2)$ time.
  - Can G be 3-colored? NP-complete (even on planar graphs)

Theorem: Any triangulation of a simple polygon can be 3-colored.

Proof: later



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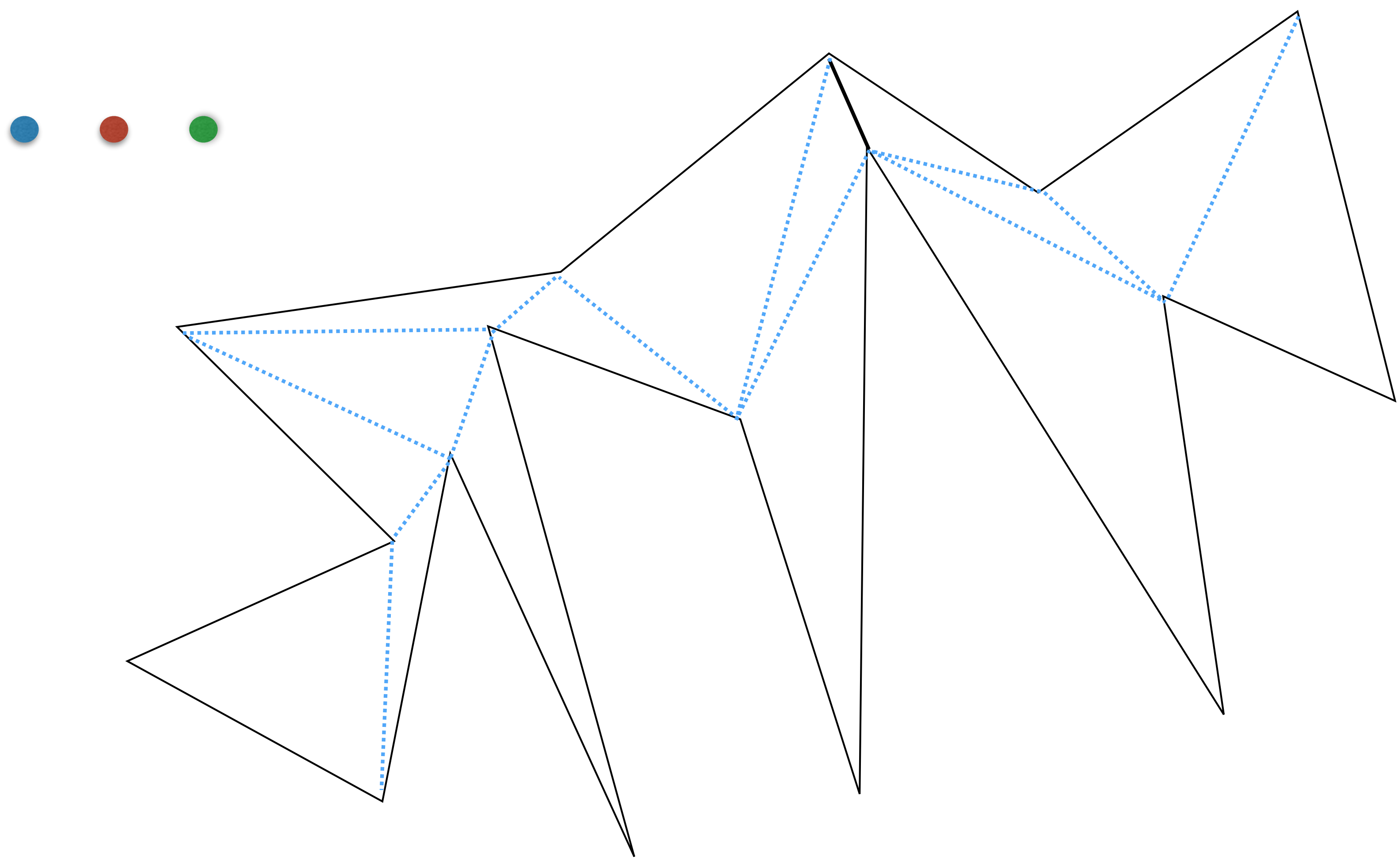

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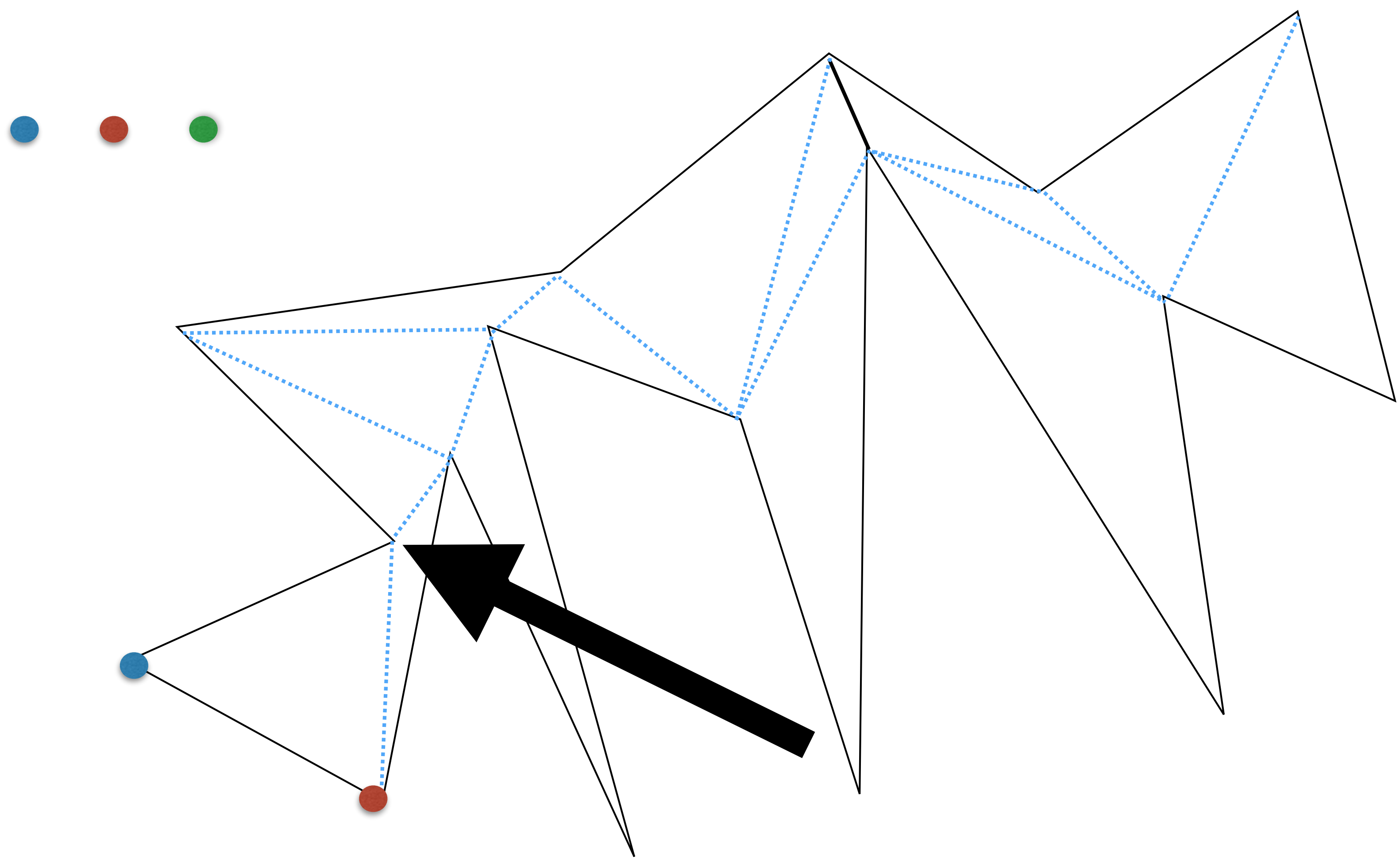

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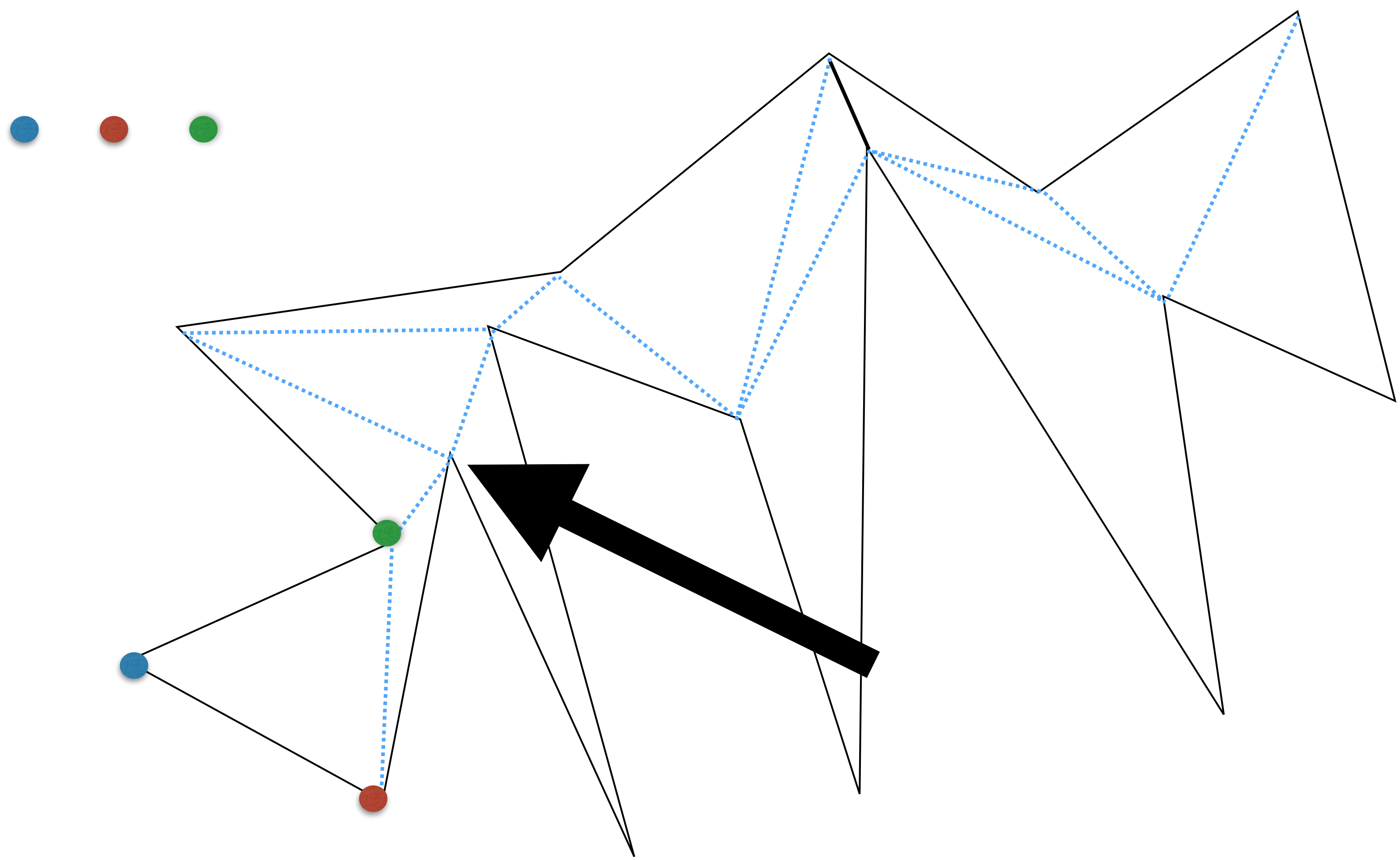
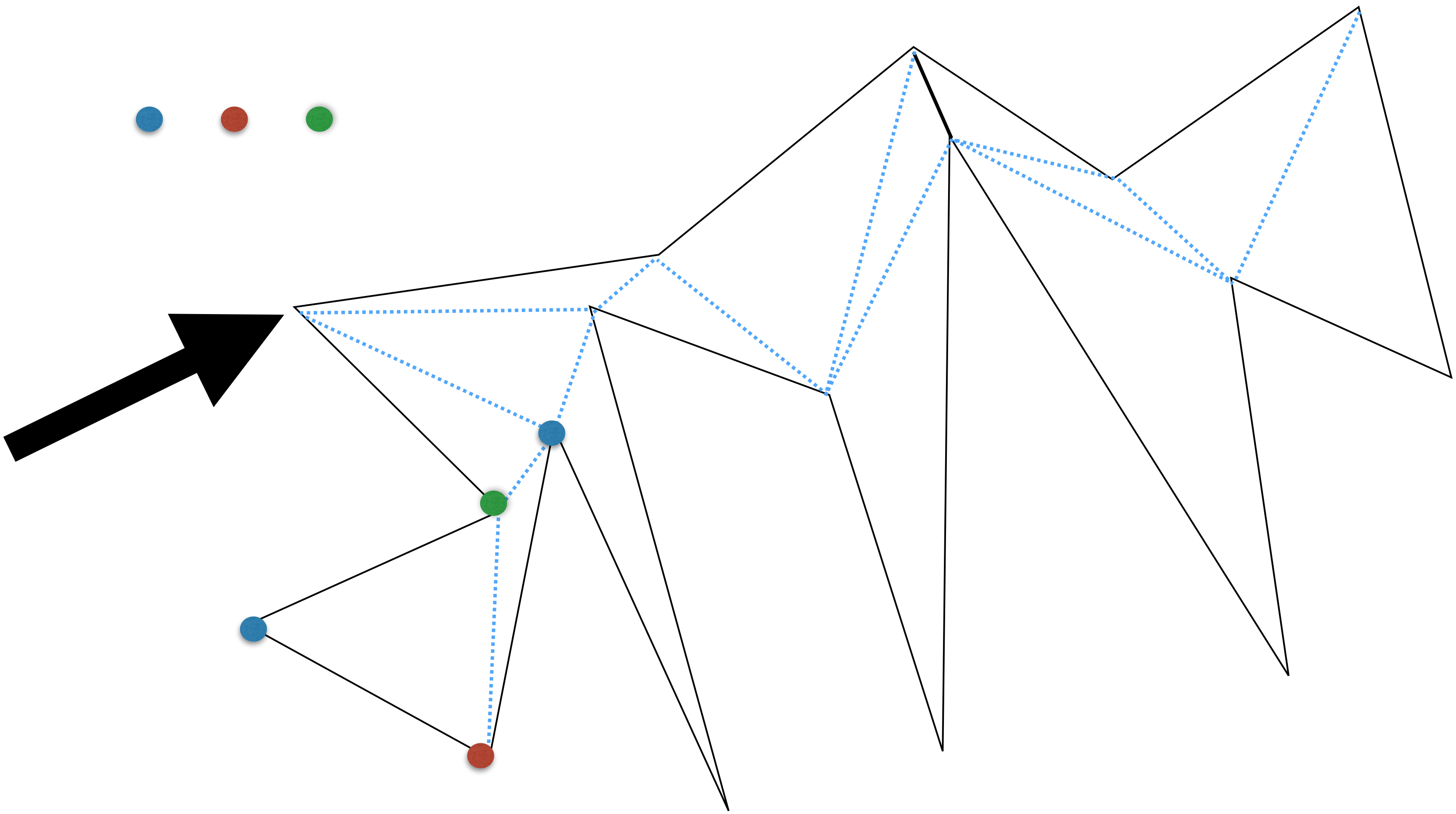
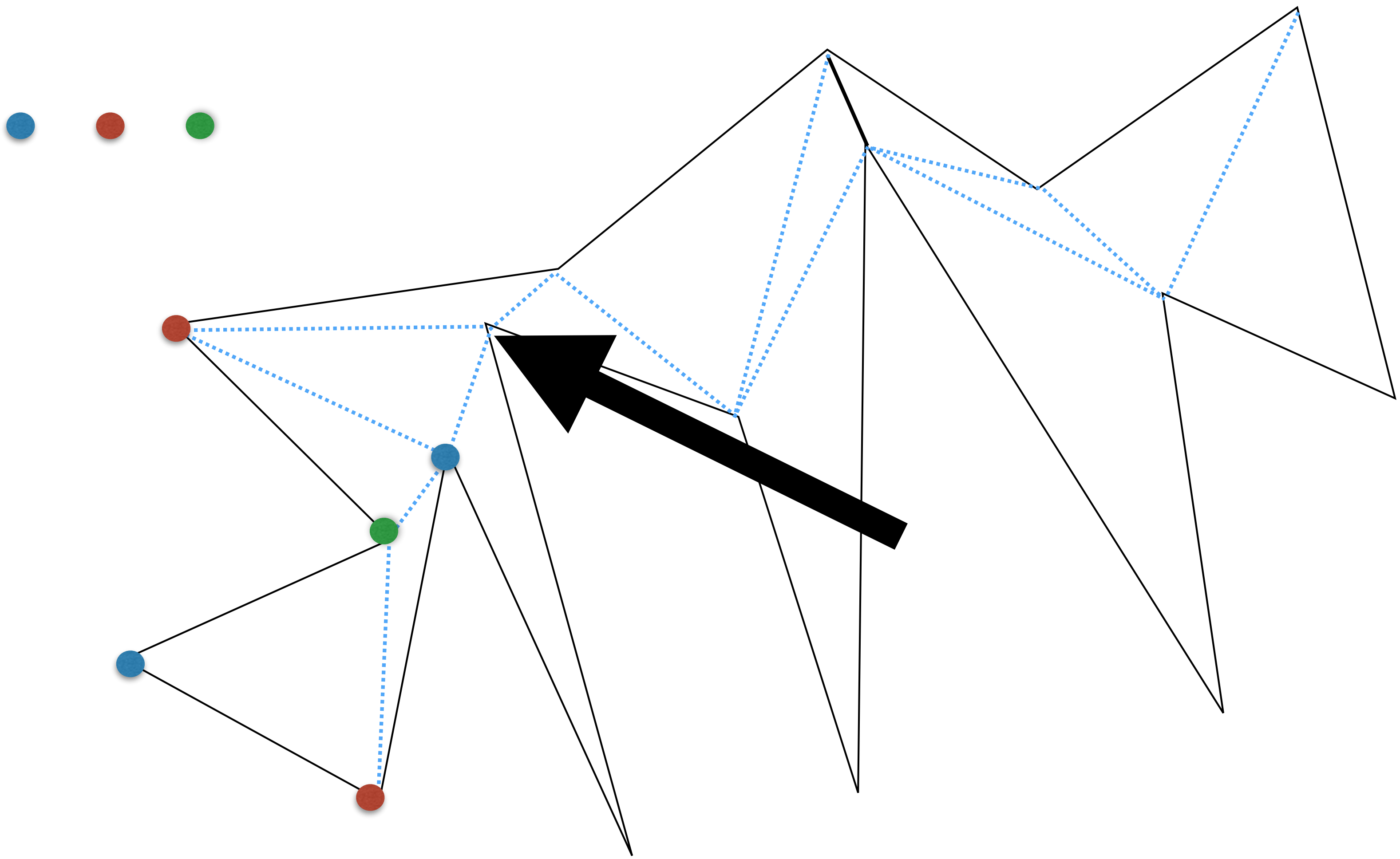
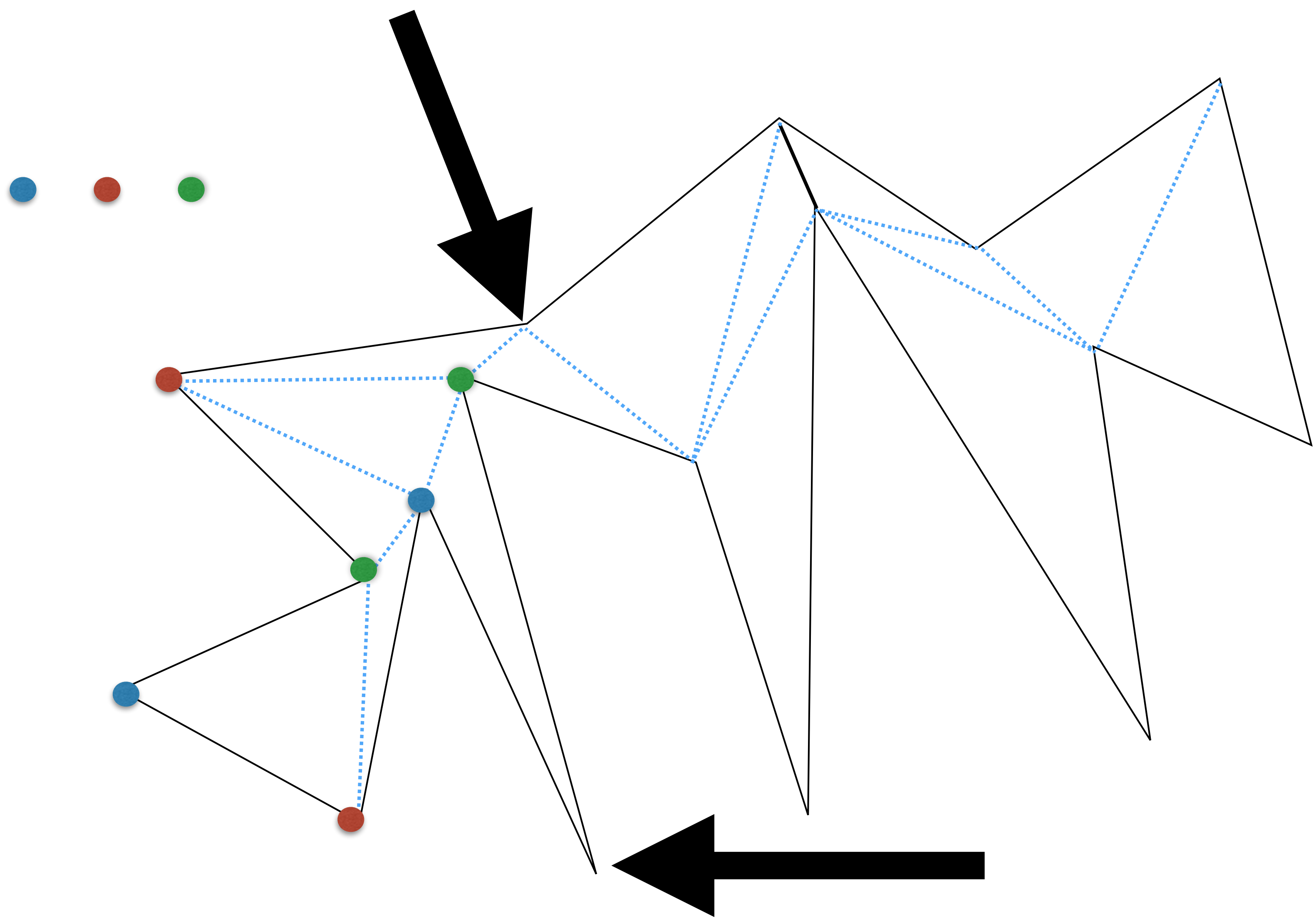
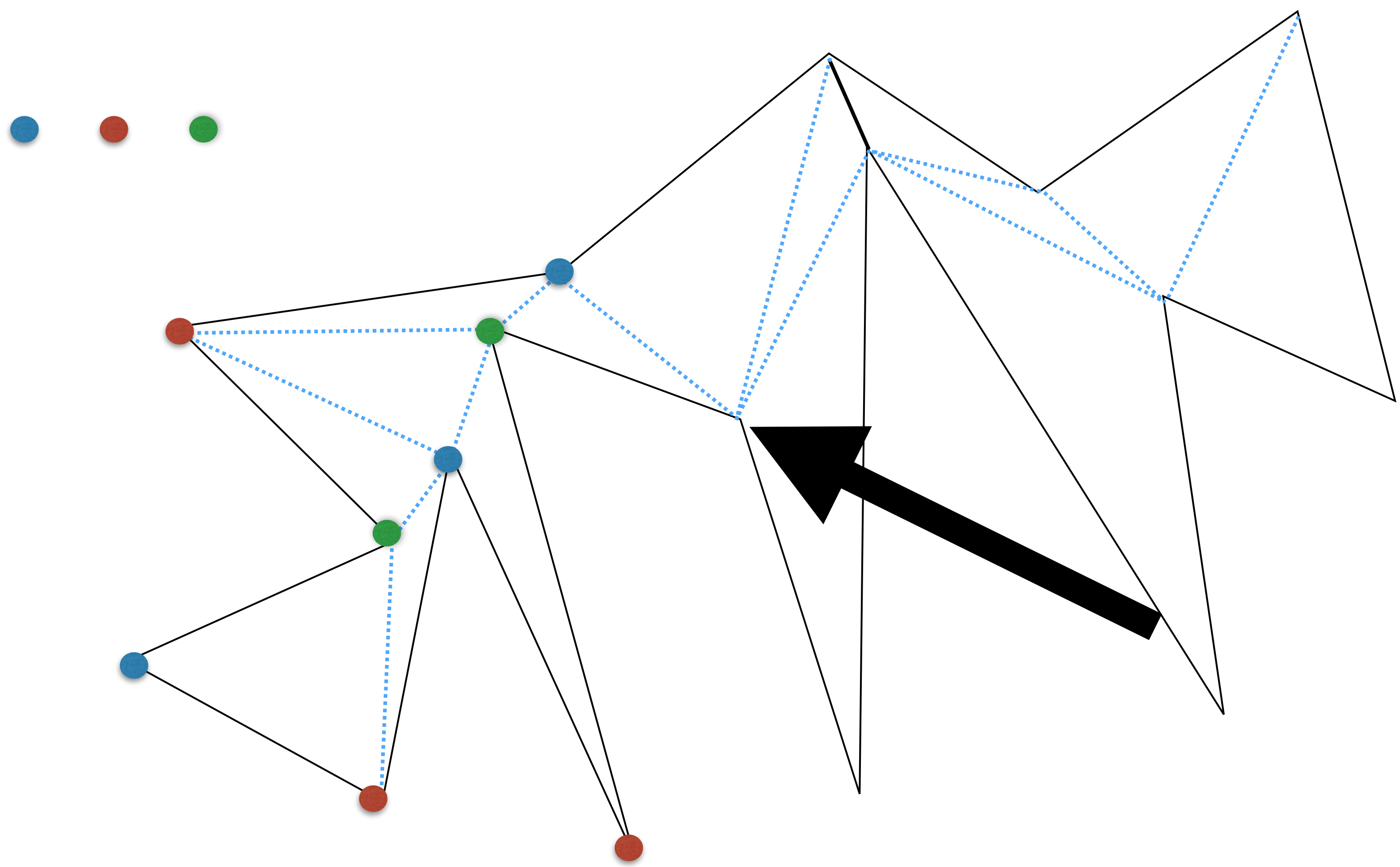
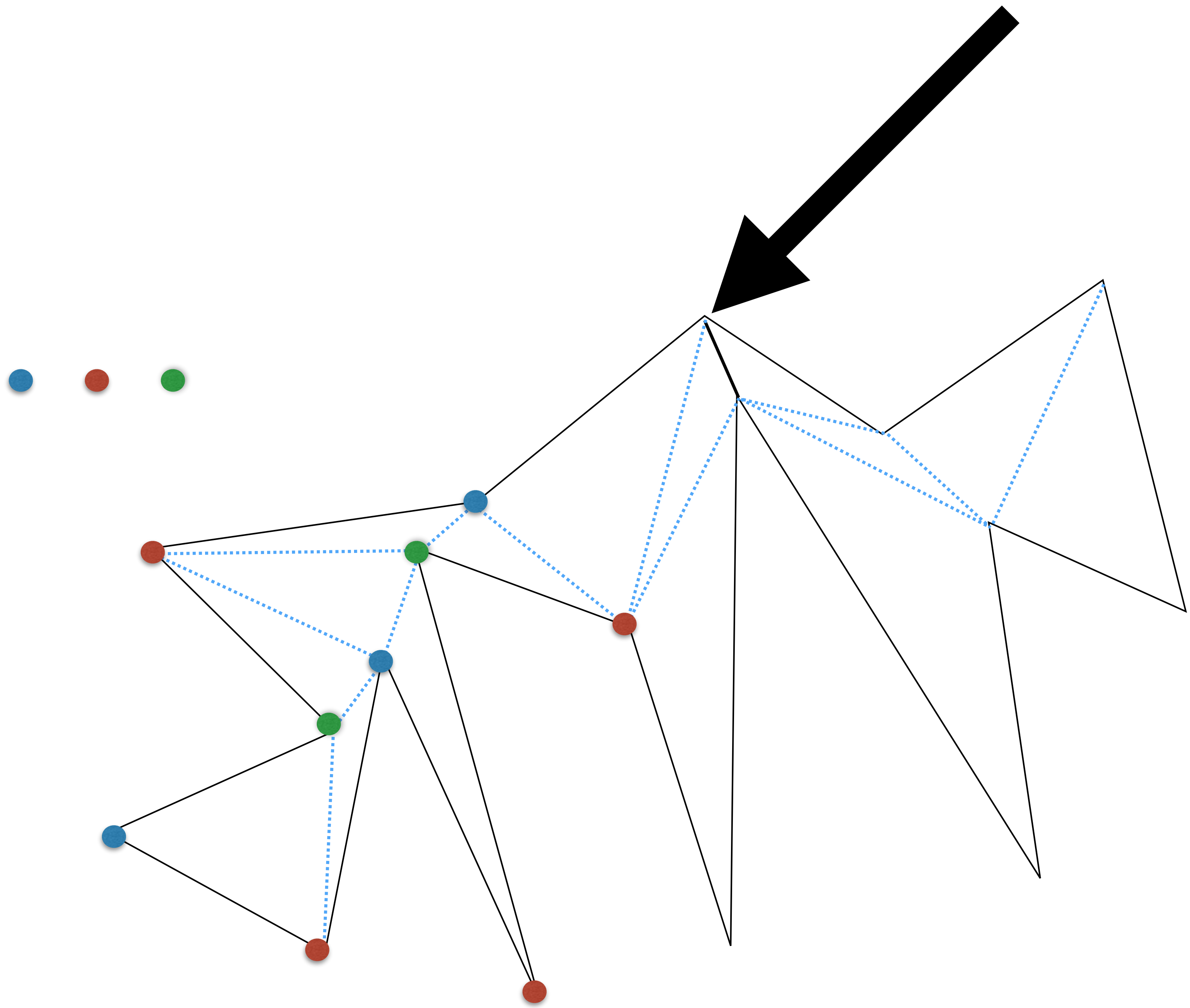


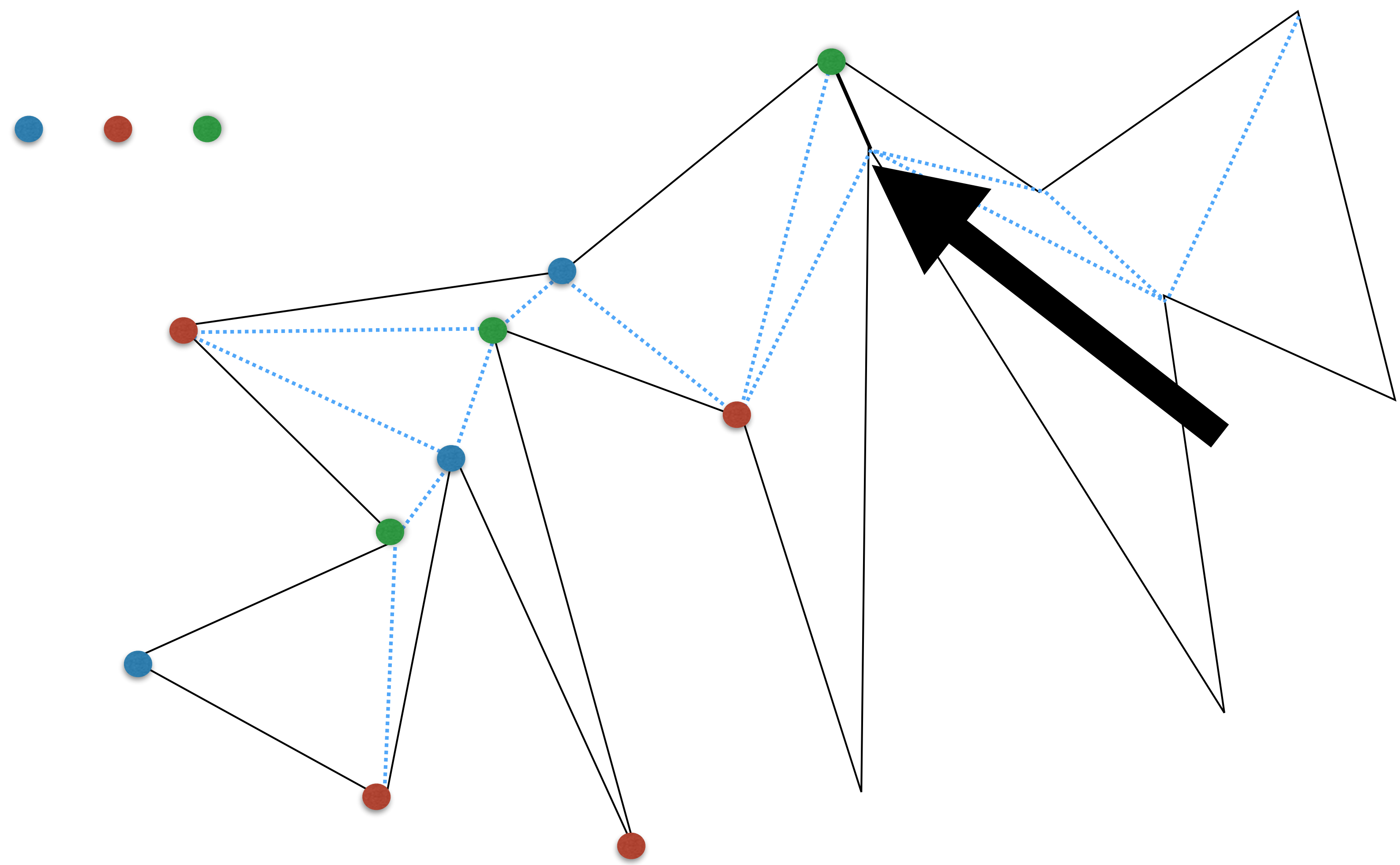


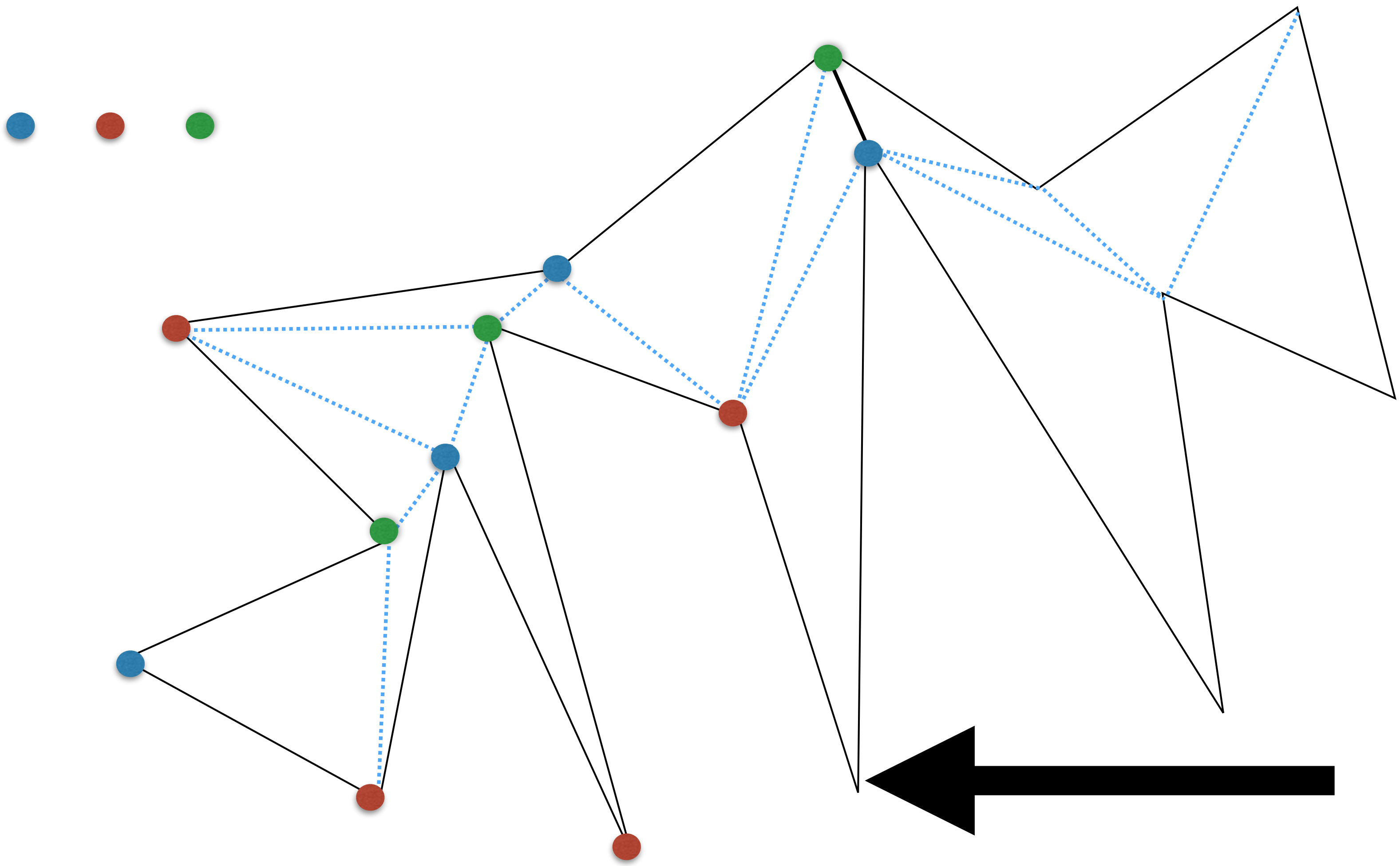


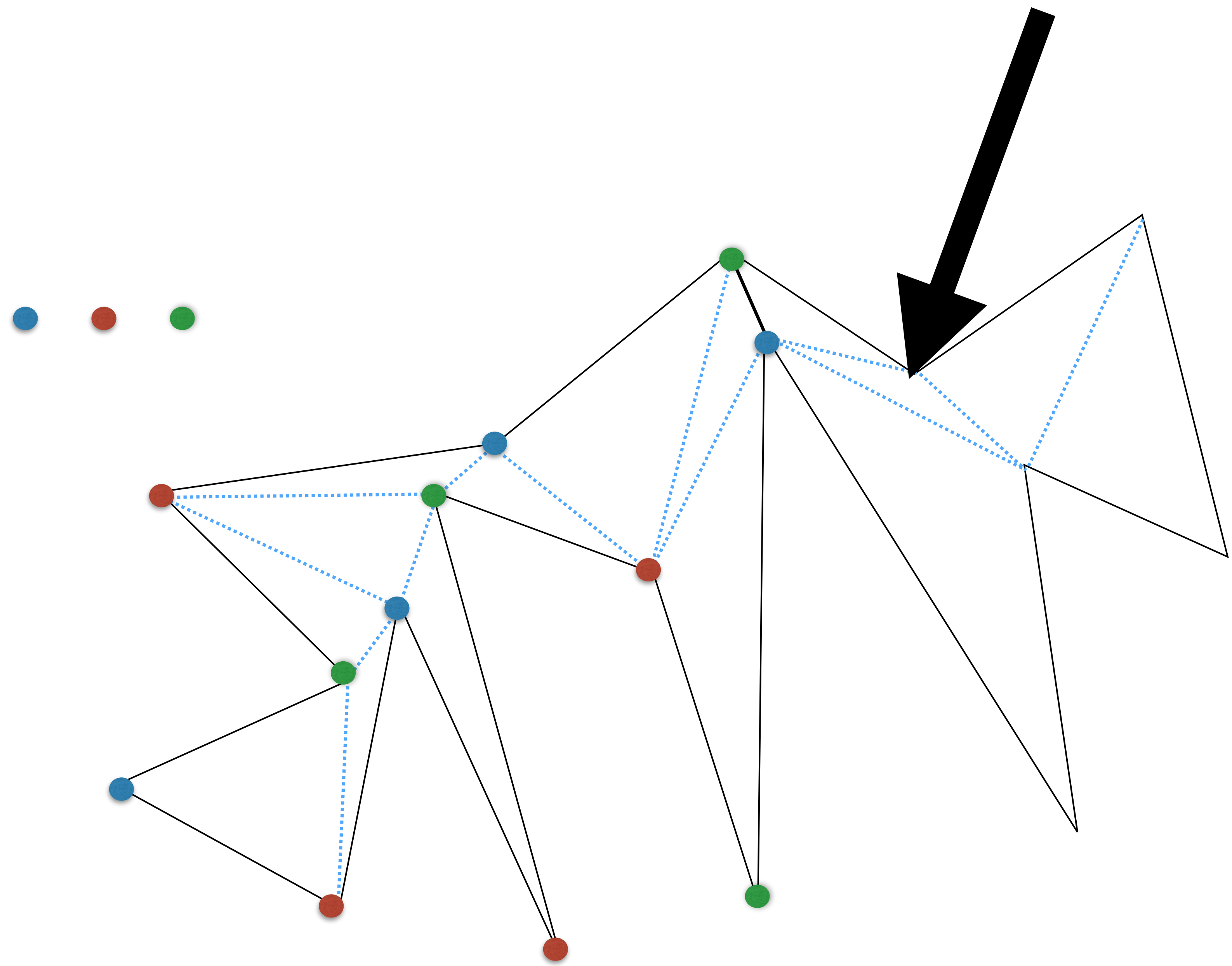


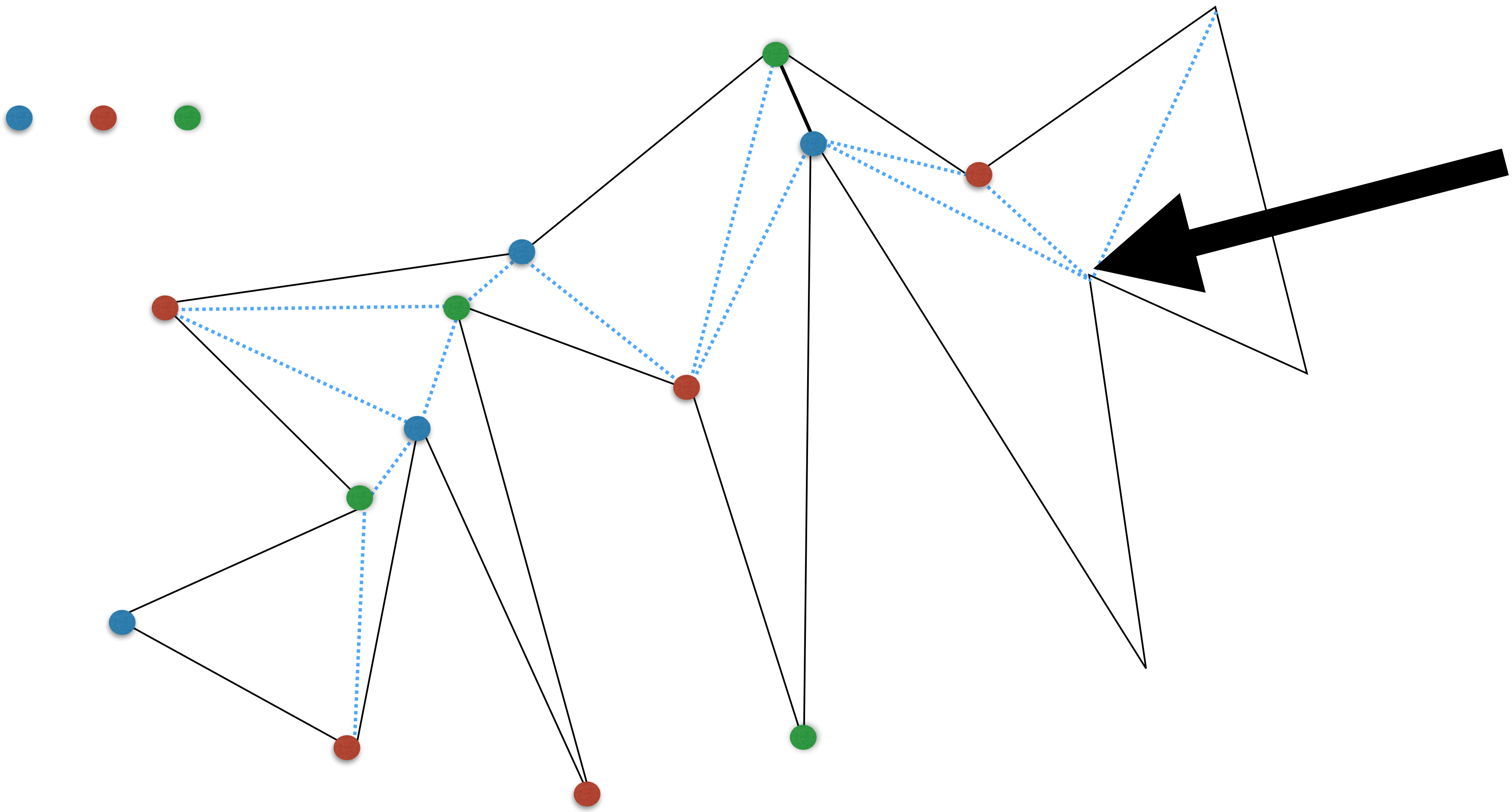


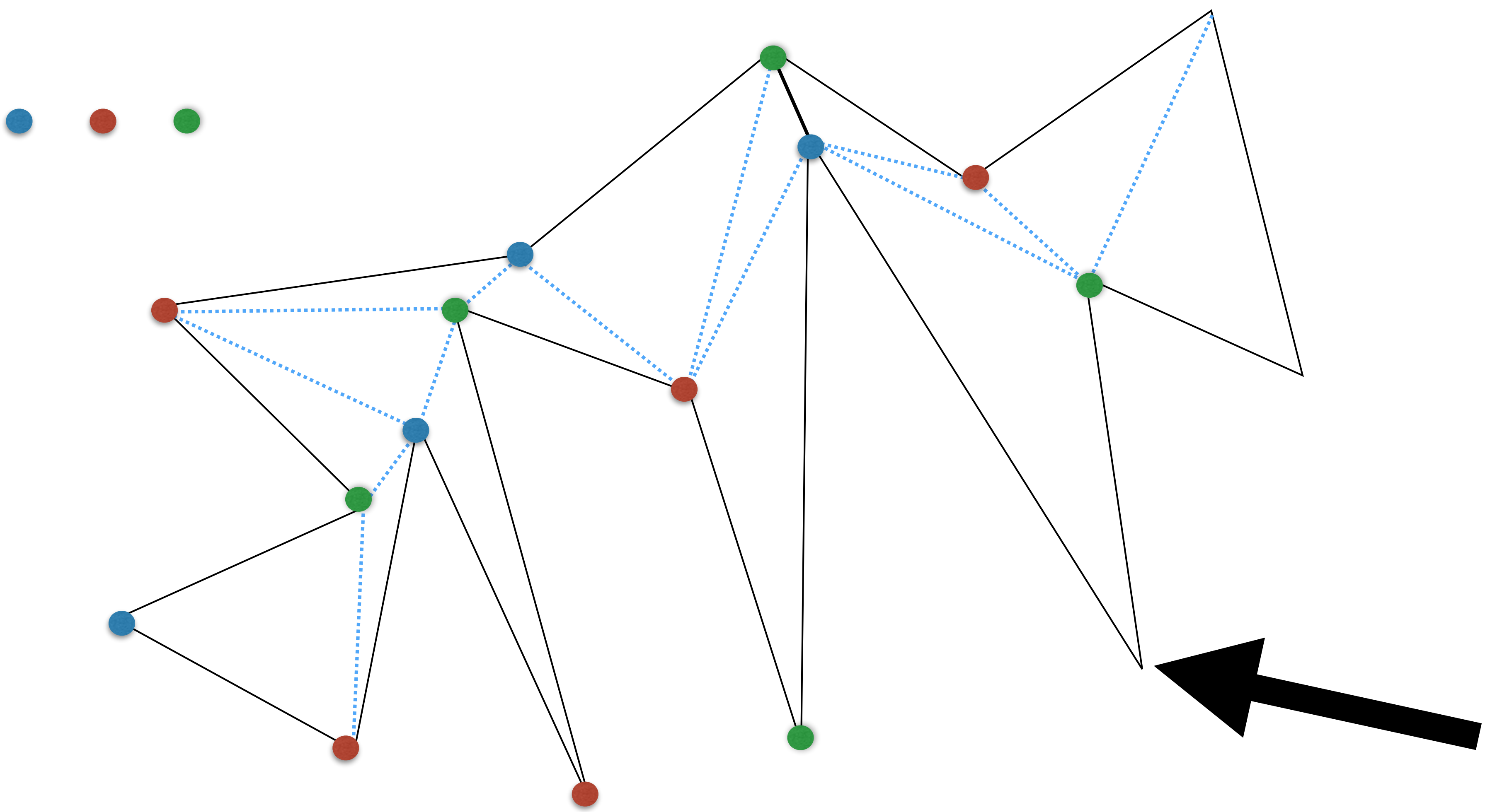


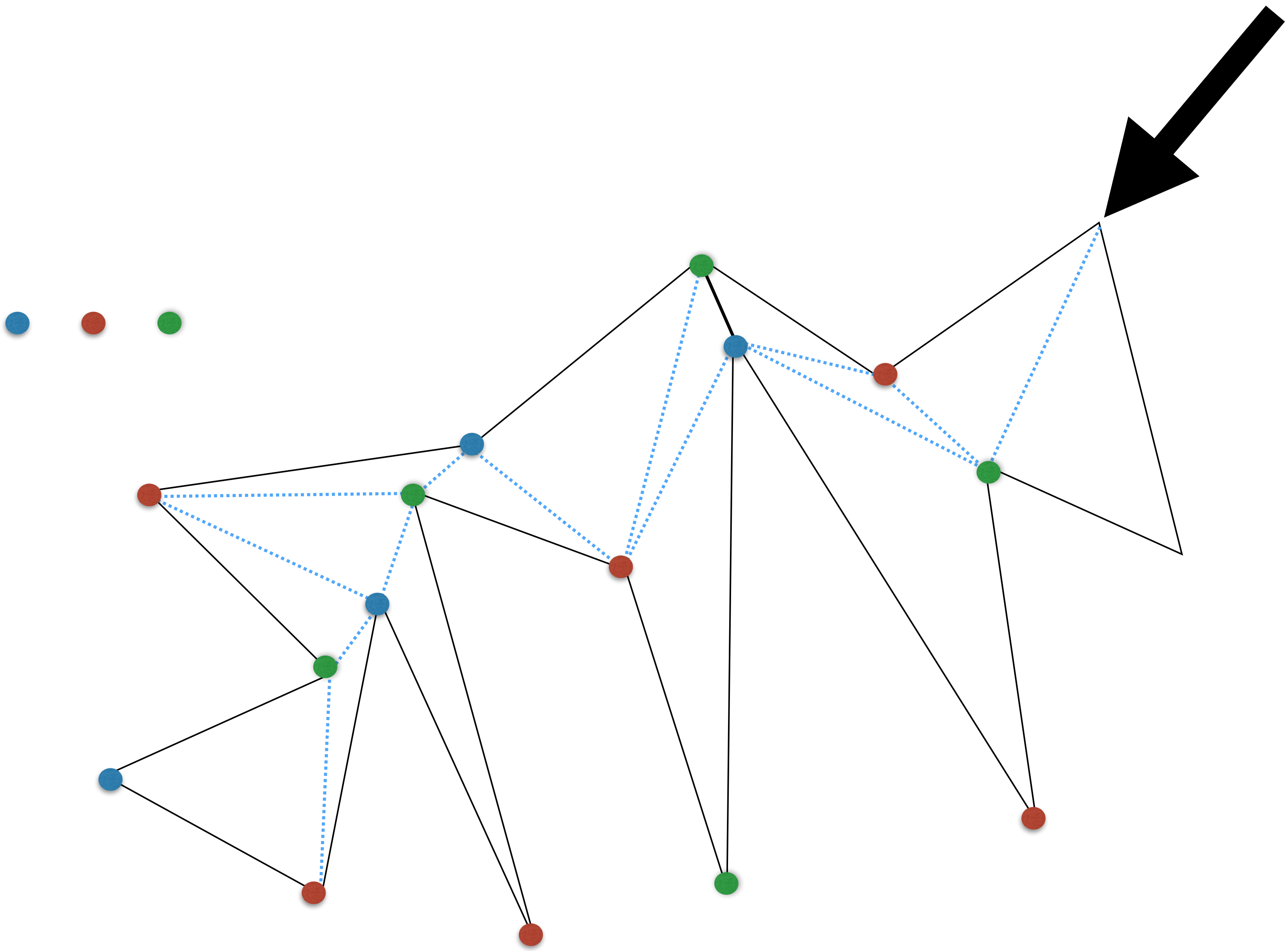


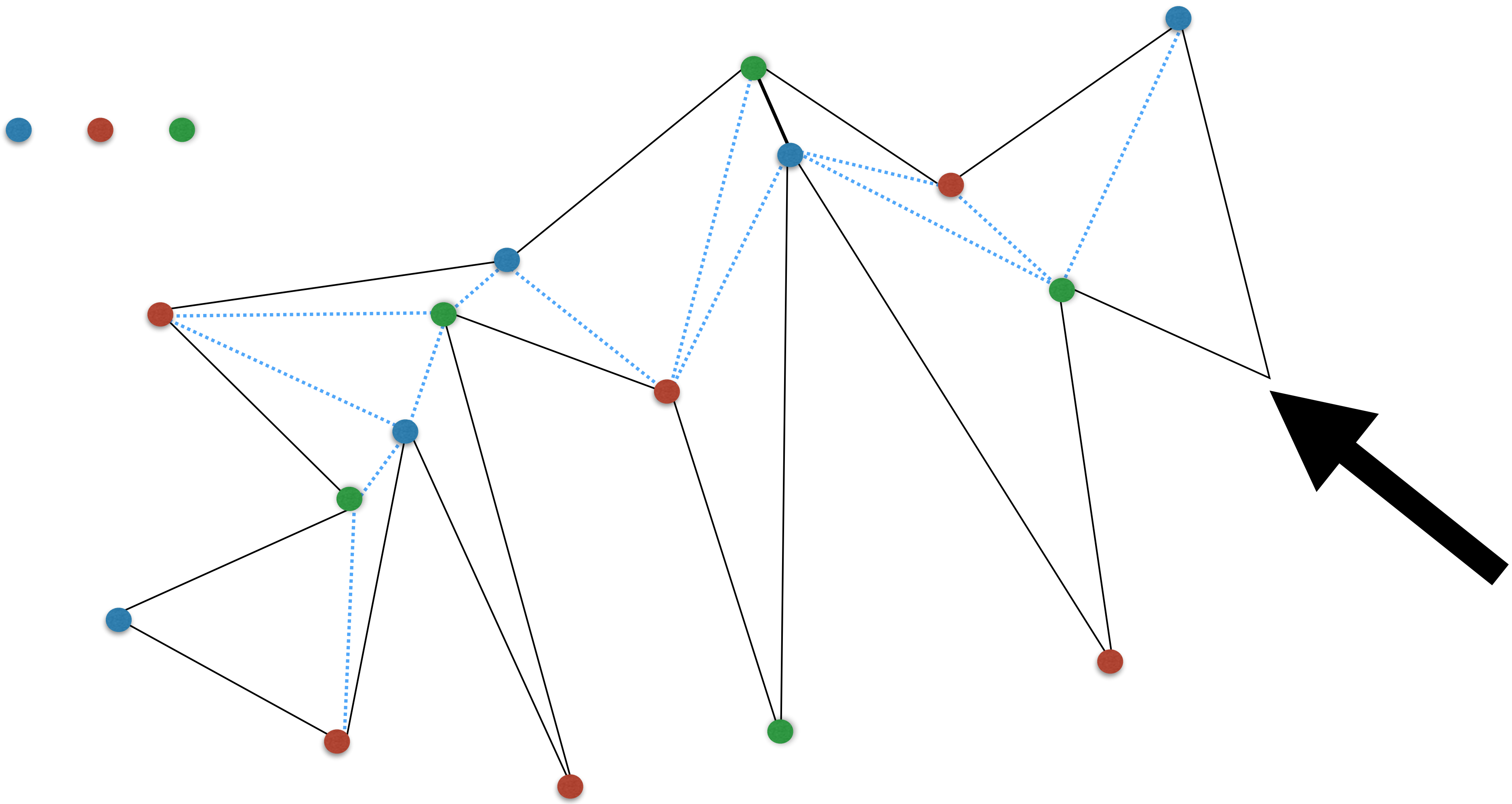


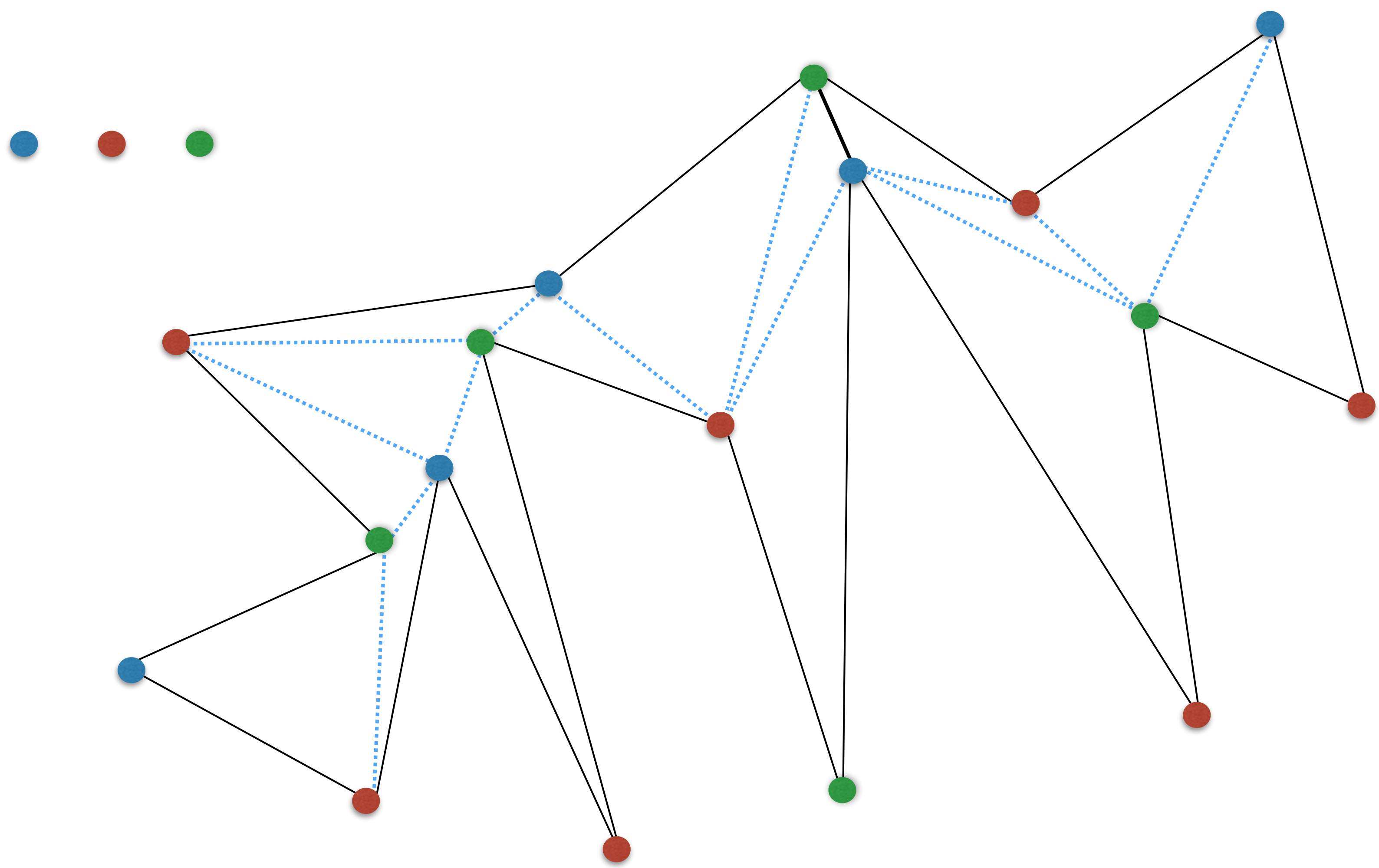






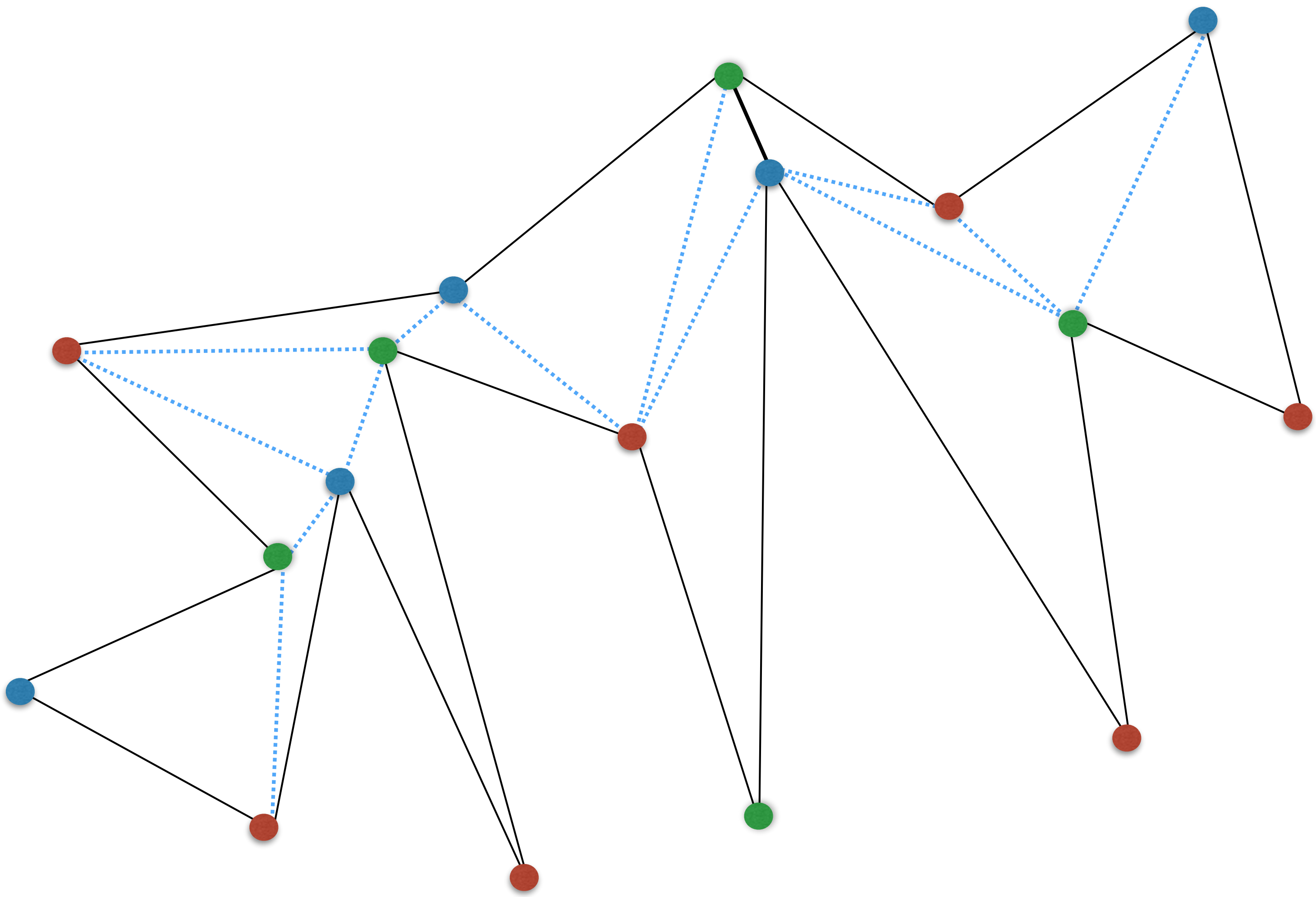






Theorem: Placing guards at vertices of one color covers P.

Proof: later



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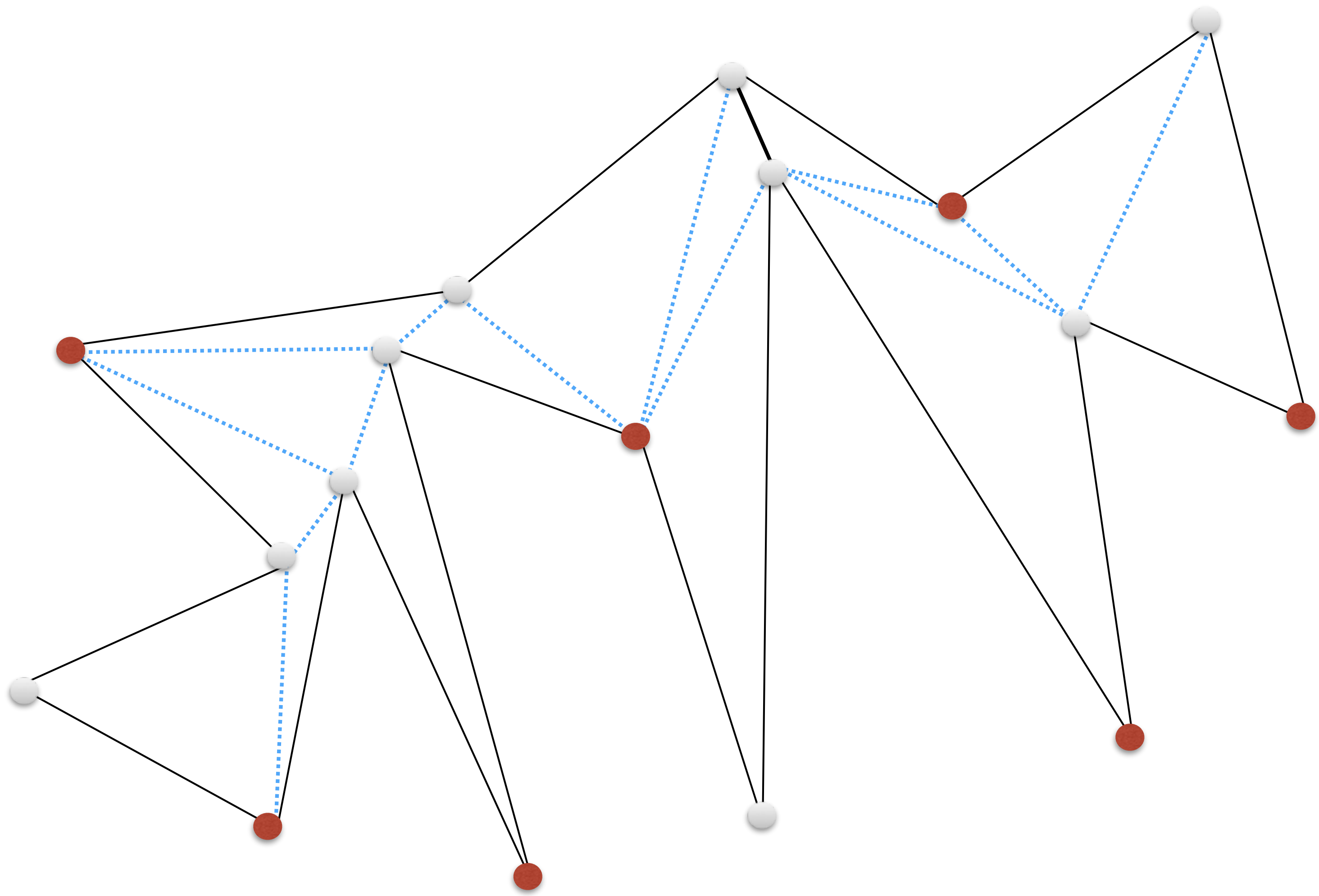
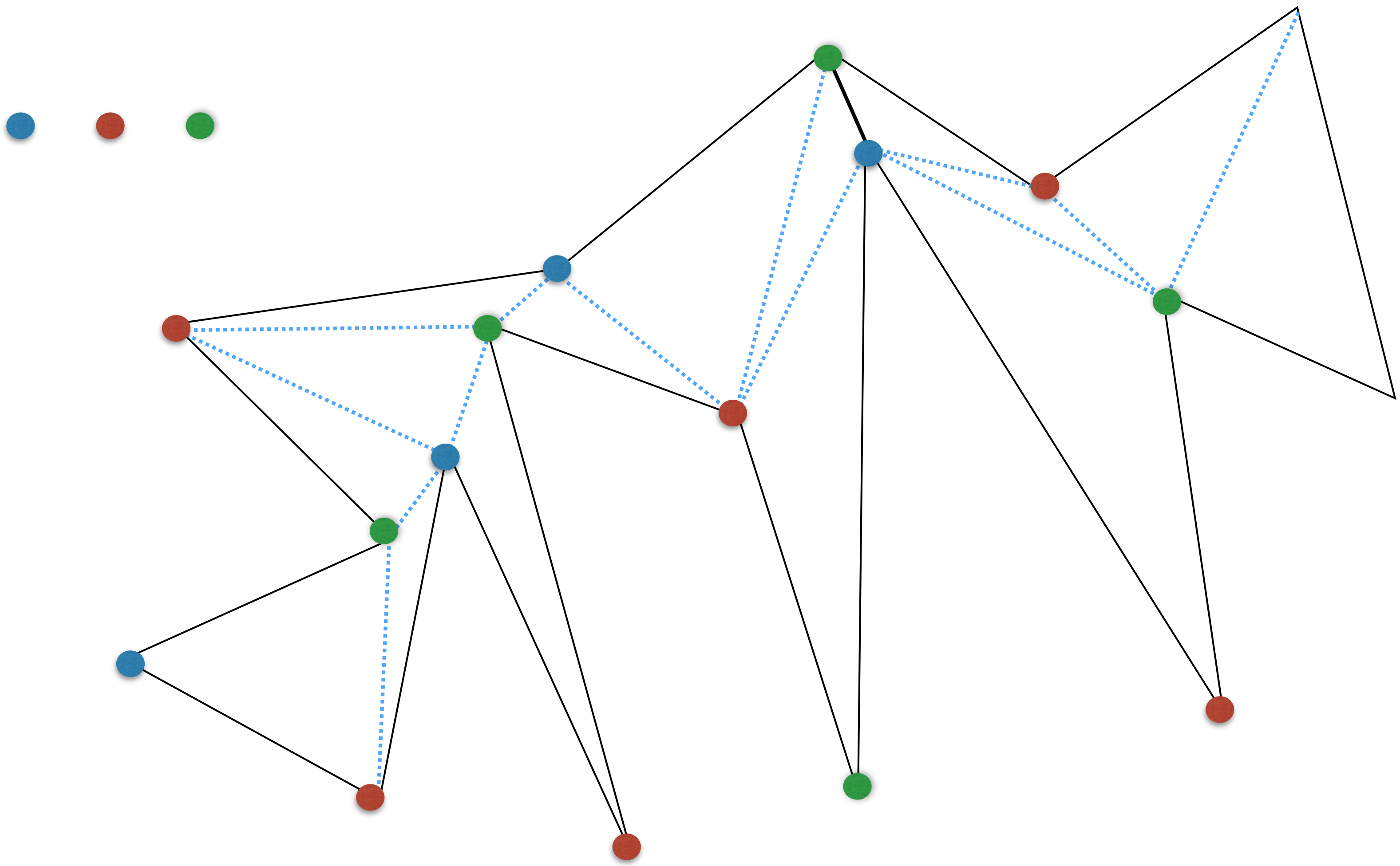


Theorem: Pick least frequent color. At most n/3 vertices of that color.

Proof: later



Theorem: Any P_n can be guarded with at most $\lfloor n/3 \rfloor$ guards.

Fisk's proof:

1. Theorem: Any polygon can be triangulated
2. Theorem: Any triangulation can be 3-colored
3. Theorem: Placing the guards at all the vertices assigned to one color guarantees the polygon is guarded.
4. Theorem: There must exist a color that's used at most $\lfloor n/3 \rfloor$ times.

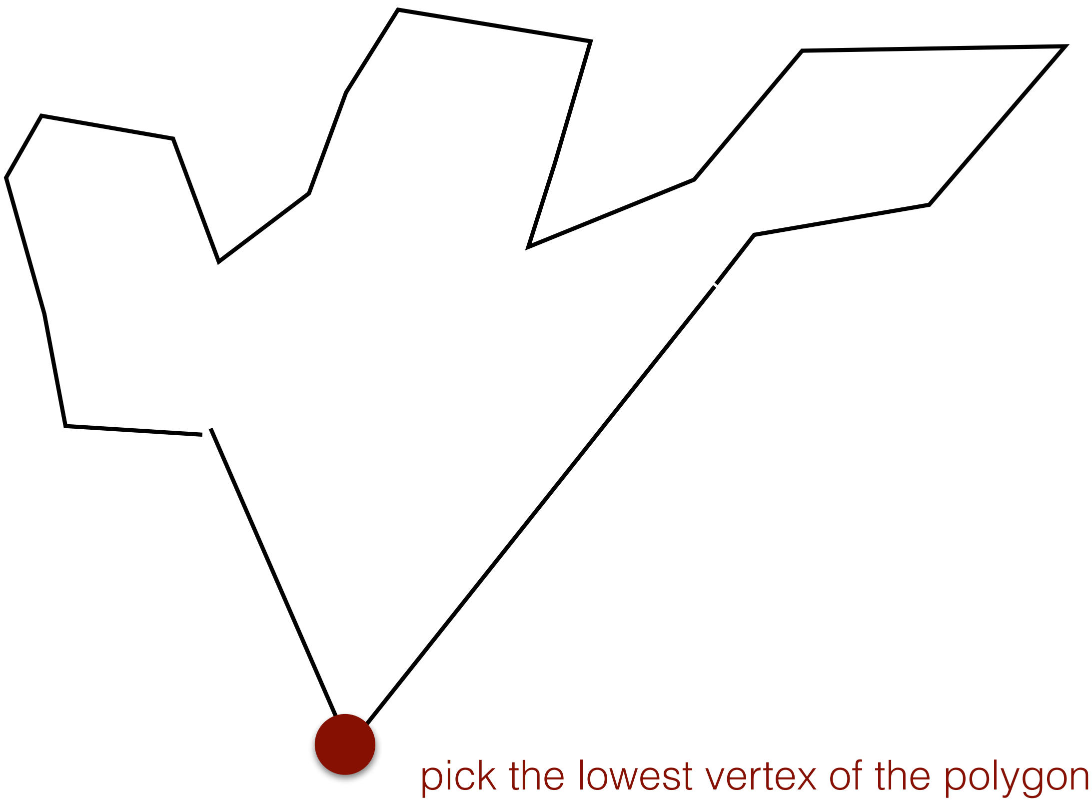
   Place guards at the vertices of that color ==> the polygon is guarded by at most $\lfloor n/3 \rfloor$ guards

The Proofs

Polygon triangulation

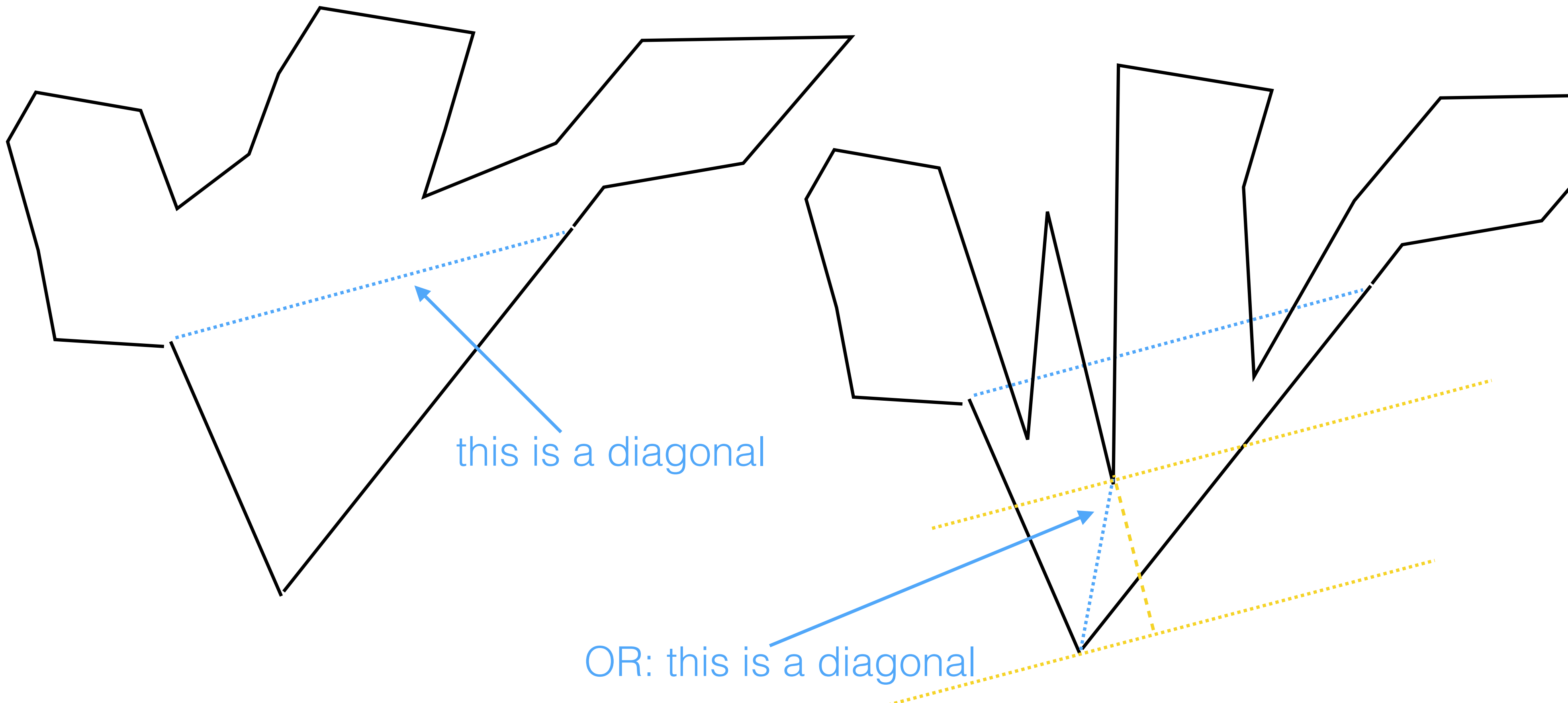
Claim 1: Any polygon contains at least one strictly convex vertex

the angle is <180



Polygon triangulation

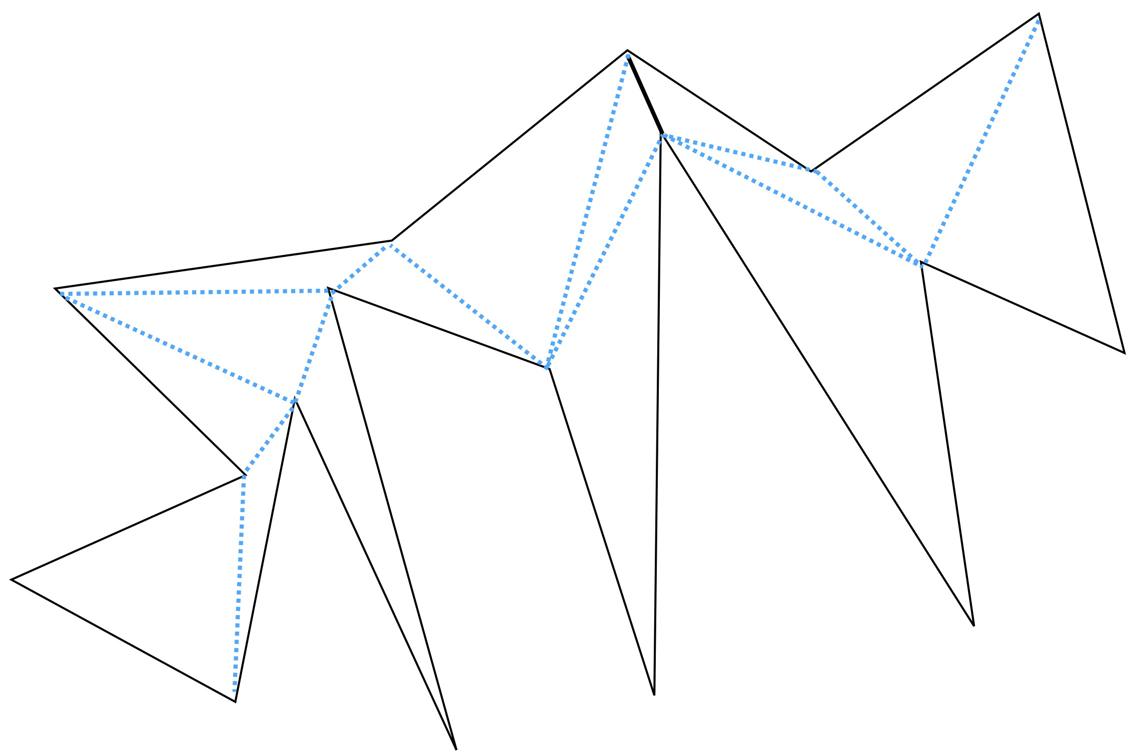
Claim 2: Any polygon with n>3 vertices has at least one diagonal.



Polygon triangulation

Theorem: Any simple polygon can be triangulated.

Proof: induction using the existence of a diagonal.



Theorem: Any triangulation of a simple polygon can be 3-colored.

Proof:

Consider a simple polygon that is triangulated and 3-colored.

Theorem: The set of red vertices guards the polygon. The set of blue vertices guards the polygon. The set of green vertices guards the polygon.

Proof:

Consider a simple polygon that is triangulated and 3-colored.

Theorem: There must exist a color that's used at most $\lfloor n/3 \rfloor$ times.

Proof: